Optimal control strategies for bistable ODE equations: Application to mosquito population replacement

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Abstract

Vector-borne diseases, in particular arboviruses, represent a major threat to human health. In the fight against these viruses, the endosymbiotic bacterium *Wolbachia* has become in recent years a promising tool as it has been shown to prevent the transmission of some of these viruses between mosquitoes and humans. In this work, we investigate optimal population replacement strategies, which consists in replacing optimally the wild population by a population carrying the aforementioned bacterium, making less likely the appearance of outbreaks of these diseases. We consider a two species model taking into account both wild and Wolbachia infected mosquitoes. To control the system, we introduce a term representing an artificial introduction of Wolbachia-infected mosquitoes. Assuming a high birth rate, we reduce the model to a simpler one regarding the proportion of infected mosquitoes. We study strategies optimizing a convex combination either of cost and time or cost and final proportion of mosquitoes in the population. We fully analyze each of the introduced problem families, proving a time monotonicity property on the proportion of infected mosquitoes and using a reformulation of the problem based on a suitable change of variable. Our results are useful in considerably more general contexts which we present.

Keywords: optimal control, Wolbachia, ordinary differential systems, epidemic vector control, bistable ODEs.

1 Introduction

1.1 Around Wolbachia control strategies

Around 700 000 people die annually due to mosquito-transmitted diseases [16]. In particular, mosquitoes of the genus Aedes, such as *Aedes Aegypti* and *Aedes Albopictus* can transmit several arboviruses as Dengue, Chikungunya, Yellow fever or Zika [9, 17]. According to the World Health Organization, 390 million people are infected by Dengue every year and 3.9 billion people in 128 countries are at risk of infection [6]. As no antiviral treatment nor efficient vaccine are known for Dengue, the current method for preventing its transmission relies mainly on targeting the vector, i.e. the mosquito [5, 4, 11]. It has been shown that the presence of the bacterium Wolbachia [10] in these mosquitoes reduces their vector capacity (capability of transmission of the associated disease) for the aforementioned arboviruses [20, 14, 19, 15]. The bacterium is transmitted from the mother to the offspring. Furthermore, there is a phenomenon called *Cytoplasmatic Incompatibility* (CI) [18, 13], which produces cross sterility between Wolbachia-infected males and uninfected females. These two key phenomena make the introduction of mosquitoes infected with Wolbachia a promising control strategy to prevent Dengue transmission.

In this work, we explore several ways of modeling optimal release strategies, in the spirit of [3], where a simpler approach involving a least squares functional was presented. We enrich the model of [3] by introducing and analyzing two relevant families of problems.

In a nutshell, we will first consider two families of functionals that are convex combinations of a term accounting for the cost of the mosquitoes used and

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- either a growing function of the time horizon, let free, but fixing the final proportion of Wolbachia-infected mosquitoes.
- or a penalization (more precisely a decreasing function) of the final proportion of Wolbachia-infected mosquitoes at the final time of the experiment. Note that the horizon of time will be considered fixed in this case.

This will lead us to introduce two large families of relevant optimization problems in order to model this issue. Analyzing them will allow us to discuss optimal strategies of mosquito releasing and also the robustness of the properties of the solutions with respect to the modeling choices (in particular the choice of the functional we optimize).

Nevertheless, the results presented in this paper are not restricted to this particular problem. In Remark 2 we state the conditions under which our results are applicable to other control problems with bistable equations.

1.2 Issues concerning modeling of control strategy

To study these issues, let us consider the same model as in [3] for modeling two interacting mosquito populations: a Wolbachia-free population n_1, and a Wolbachia carrying one, n_2. The resulting system reads

$$\left\{\begin{array}{l} \frac{dn_1(t)}{dt} = b_1 n_1(t)\left(1 - s_h \frac{n_2(t)}{n_1(t)+n_2(t)}\right)\left(1 - \frac{n_1(t)+n_2(t)}{K}\right) - d_1 n_1(t), \\ \frac{dn_2(t)}{dt} = b_2 n_2(t)\left(1 - \frac{n_1(t)+n_2(t)}{K}\right) - d_2 n_2(t) + u(t)\ ,\ t>0, \\ n_1(0) = n_1^0\ ,\ n_2(0) = n_2^0, \end{array}\right. \tag{1}$$

where

- the parameter $s_h \in [0,1]$ is the cytoplasmic incompatibility (CI) rate[1].
- The other parameters (b_i, d_i) for $i \in \{1,2\}$ are positive and denote respectively the intrinsic mortality and intrinsic birth rates. Moreover, we assume that $b_i > d_i$, $i = 1,2$.
- $K > 0$ denotes the environmental carrying capacity. Note that the term $(1 - s_h \frac{n_2}{n_1+n_2})$ models the CI.
- $u(\cdot) \in L^\infty(\mathbb{R}_+)$ plays the role of a control function that we will use to act upon the system. This control function represents the rate at which Wolbachia-infected mosquitoes are introduced into the population.

System (1) for modeling mosquito population dynamics with *Wolbachia* has been first introduced in [7, 8]. We also mention [12] where this model is coupled with an epidemiological one.

The aim of this technique is to replace the wild population by a population of Wolbachia-infected mosquitoes. To understand mathematically this question, it is important to recall that, under the additional assumption

$$1 - s_h < \frac{d_1 b_2}{d_2 b_1} < 1 \tag{2}$$

satisfied in practice [3], System (1) has four non-negative steady states, among which two which are locally asymptotically stable, namely:

$$\bar{\mathbf{n}}_1 = (n_1^*, 0) := \left(K\left(1 - \frac{d_1}{b_1}\right), 0\right) \quad \text{and} \quad \bar{\mathbf{n}}_2 = (0, n_2^*) := \left(0, K\left(1 - \frac{d_2}{b_2}\right)\right).$$

Observe that $\bar{\mathbf{n}}_1$ corresponds to a mosquito population without Wolbachia-infected individuals whereas $\bar{\mathbf{n}}_2$ corresponds to a mosquito population composed exclusively of infected individuals. Note that the two remaining steady-states are unstable: they correspond to the whole population extinction and a coexistence state.

Hence, the optimal control issue related to the mosquito population replacement problem can be recast as:

[1] Indeed, when $s_h = 1$, CI is perfect, whereas when $s_h = 0$ there is no CI

Starting from the equilibrium $\bar{\mathbf{n}}_1$, how to design a control steering the system as close as possible to the equilibrium state $\bar{\mathbf{n}}_2$, minimizing at the same time the cost of the releases?

Of course, although this is the general objective we wish to pursue, the previous formulation remains imprecise and it is necessary to clarify what is meant by "the cost of release" and the set in which it is relevant to choose the control function.

Following [2] and [3], we will impose several biological constraints on the control function u: the rate at which mosquitoes can instantaneously be released will be assumed bounded above by some positive constant M, and so will be the total amount of released infected mosquitoes up to the final time T. The set of admissible control functions $u(\cdot)$ thus reads

$$\mathcal{U}_{T,C,M} := \left\{ u \in L^\infty(0,T), 0 \leqslant u \leqslant M \text{ a.e. in } (0,T), \int_0^T u(t)dt \leqslant C \right\}. \tag{3}$$

As shown in [3], System (1) can be reduced to a single equation under the hypothesis of high birth rates, i.e. considering $b_1 = b_1^0/\varepsilon$, $b_2 = b_2^0/\varepsilon$ and letting ε decrease to 0. In this frame, the proportion $n_2/(n_1+n_2)$ of Wolbachia-infected mosquitoes in the population, uniformly converges to p, the solution of a simple scalar ODE, namely

$$\left\{ \begin{array}{l} \frac{dp}{dt}(t) = f(p(t)) + u(t)g(p(t)), \quad t \in (0,T) \\ p(0) = 0, \end{array} \right. \tag{4}$$

where

$$f(p) = p(1-p)\frac{d_1 b_2^0 - d_2 b_1^0(1-s_h p)}{b_1^0(1-p)(1-s_h p) + b_2^0 p} \qquad \text{and} \qquad g(p) = \frac{1}{K}\frac{b_1^0(1-p)(1-s_h p)}{b_1^0(1-p)(1-s_h p) + b_2^0 p}.$$

We remark that $f(0) = f(1) = 0$ and, under assumption (2), there exists a single root of f strictly between 0 and 1 at $p = \theta = \frac{1}{s_h}\left(1 - \frac{d_1 b_2^0}{d_2 b_1^0}\right)$. The function $p \mapsto g(p)$ is non-negative, strictly decreasing in $[0,1]$ and such that $g(1) = 0$.

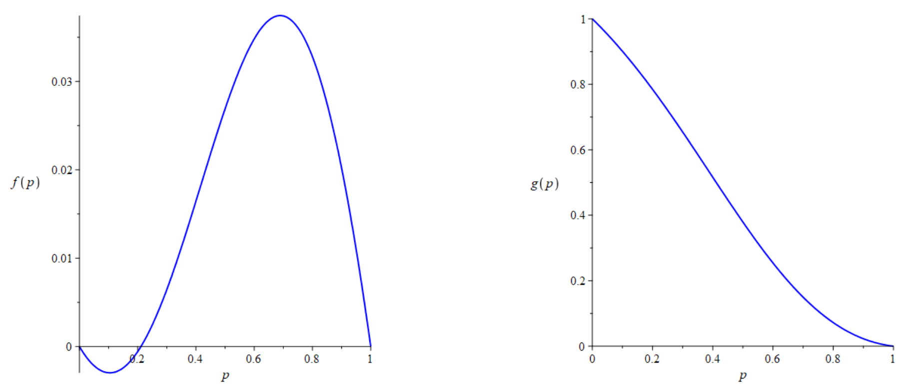

Figure 1: Plots of $p \mapsto f(p)$ (left) and $p \mapsto g(p)$ (right) for the values of the parameters in Table 1. In this case $\theta \approx 0.211$.

In the absence of a control function, the equation on p simplifies into $\frac{dp}{dt} = f(p)$. This is a bistable system, with an unstable equilibrium at $p = \theta$ and two stable equilibria at $p = 0$ and $p = 1$. Notice that the derivative of the function f/g has a unique zero p^* in $(0,\theta)$ defined by

$$p^* = \frac{1}{s_h}\left(1 - \sqrt{\frac{d_1 b_2^0}{d_2 b_1^0}}\right), \tag{5}$$

which will be useful in the following.

In [3], the control problem

$$\inf_{u \in \mathcal{U}_{T,C,M}} J(u), \quad \text{with } J(u) = \frac{1}{2}n_1(T)^2 + \frac{1}{2}\left[(n_2^* - n_2(T))_+\right]^2. \tag{6}$$

related to the aforementioned system (1), is considered. Denoting by $J^\varepsilon(u)$ the criterion $J(u)$ where the birth rates b_1 and b_2 have been respectively replaced by $b_{1,\varepsilon} = b_1^0/\varepsilon$ and $b_{2,\varepsilon} = b_2^0/\varepsilon$, with $\varepsilon > 0$, a Γ-convergence type result is proven [3, Proposition 2]. More precisely, any solution u_ε of Problem (6) with birth rates $b_{1,\varepsilon}$ and $b_{2,\varepsilon}$ converges weakly-star in $L^\infty(0,T)$ to a solution of the reduced problem (4). Moreover,

$$\lim_{\varepsilon\to 0} \inf_{u\in\mathcal{U}_{T,C,M}} J^\varepsilon(u) = \inf_{u\in\mathcal{U}_{T,C,M}} J^0(u),$$

where

$$J^0(u) = \lim_{\varepsilon\to 0} J^\varepsilon(u) = K(1-p(T))^2 \tag{7}$$

and p is the solution of (4) associated to the control function choice $u(\cdot)$. The arguments exposed in [3] can be adapted easily to our problem. Since the solutions of both the full problem (6) and the minimization of J^0 given by (7) will be close in the sense above, it is relevant to investigate the later, which is easier to study both analytically and numerically.

We now introduce the two families of optimal control problems we will consider in the following sections. Although the model (4) driving the evolution of the Wolbachia-infected mosquitoes density is the same as in [3], we will enrich it by introducing and analyzing new families of problems in which

- the horizon of time can be let free;
- the cost of producing Wolbachia-infected mosquitoes can be included. Since such a cost is not so easy to take into account, we will write it in a rather general way

$$\int_0^T j_1(u(t))\,dt \tag{8}$$

where $j_1 : \mathbb{R} \to \mathbb{R}$ denotes a increasing function such that $j_1(0) = 0$.

To take the time of the experiment and the final state into account in the cost functional, we will use a function $j_2 : \mathbb{R}_+ \times [0,1] \ni (T,p)) \mapsto j_2(T,p) \in \mathbb{R}$.

Let us now present the two families of problems we will deal with. We will be led to make the following assumptions, in accordance with the modelling above:

$$\left\{\begin{array}{l} j_1(\cdot) \text{ is a non-negative increasing function such that } j_1(0)=0, \text{ two times differentiable,} \\ \text{either strictly concave, linear or strictly convex on } (0,T). \\ \\ j_2(\cdot) \text{ is a non-negative function of class } \mathcal{C}^1, \text{ strictly increasing w.r.t. its first variable} \\ \text{and strictly decreasing w.r.t. its second variable. Moreover, for all } p\in[0,1], \\ \lim_{T\to+\infty} j_2(T,p) = +\infty. \end{array}\right. \tag{$\mathcal{H}$}$$

Family 1

A first way of modeling optimal strategy for releasing Wolbachia-infected mosquitoes consists in minimizing a convex combination of the time horizon, denoted T, and the cost of producing and releasing the mosquitoes defined by (8), by imposing a target value on the final density of Wolbachia-infected mosquitoes. This leads to introduce the following optimal control problem

$$\left\{\begin{array}{l} \inf_{\substack{u\in\mathcal{U}_{T,C,M}\\ T>0}} J_\alpha(T,u), \\ p' = f(p) + ug(p) \text{ in } (0,T),\ p(0)=0\ ,\ p(T)=p_T\ , \end{array}\right. \tag{$\mathcal{P}^{1,\alpha}_{p_T,C,M}$}$$

where $p_T \in (0,1)$ is given and $J_\alpha(u)$ is defined by

$$J_\alpha(T,u) = (1-\alpha)\int_0^T j_1(u(t))dt + \alpha j_2(T,p(T)), \tag{9}$$

where $\alpha\in[0,1]$, $j_1(\cdot)$ and $j_2(\cdot)$ satisfy ($\mathcal{H}$) and $\mathcal{U}_{T,C,M}$ is given by (3). The function $(T,p)\mapsto j_2(T,p)$ aims at penalizing the time used in our case. Once the existence of solutions is established, it will be fixed to be $j_2(T,p(T)) = T$. In what follows, we will not tackle the case where $\alpha = 0$ since in that case, existence may not be guaranteed. More precisely, it is rather easy to show that in that case, Problem ($\mathcal{Q}^{1,\alpha}_{p_T,M}$) has no solution whenever $p_T > \theta$.

$\boxed{\text{Family 2}}$

Another possible way of modeling optimal strategy for releasing Wolbachia-infected mosquitoes consists in minimizing a convex combination of the final distance from $p(T)$ to the state of total invasion $p = 1$ and the cost of producing and releasing the mosquitoes defined by (8). In that case, we fix the horizon of time T and let $p(T)$ free. This leads to consider the problem

$$\begin{cases} \inf\limits_{u \in \mathcal{U}_{T,C,M}} J_\alpha(u), \\ p' = f(p) + ug(p), \ p(0) = 0 \ , \end{cases} \qquad (\mathcal{P}^{2,\alpha}_{T,C,M})$$

where $\alpha \in [0, 1]$, $j_1(\cdot)$ and $j_2(\cdot)$ satisfy $(\mathcal{H})$ and $\mathcal{U}_{T,C,M}$ is given by (3). The main difference here with respect to the previous case is the fact that the time horizon T is fixed, $p(T)$ is free and that $j_2(T, p(T))$ now represents a function penalizing the final distance to a certain final state (typically, the state of total invasion $p = 1$). Since T in this family is fixed, abusing of the notation we will write $J_\alpha(u)$ instead of $J_\alpha(T, u)$, but $J_\alpha(u)$ will still be defined by (9). After establishing the existence of solutions to this problem we will fix $j_2(T, p(T)) = (1 - p(T))^2$ as in (7). A study of similar problems in a much more limited setting can be found in [1].

1.3 Main results

Let us state here briefly the main results of this work. These results will be further detailed in sections 2.2 and 3.1 respectively. In this section, in order to avoid too much technicality, we provide simplified statements of the main contributions of this article. Let us fix $M > 0$ and $C > 0$, and let us consider $j_1(\cdot)$ satisfying the hypothesis stated above in $(\mathcal{H})$.

Our first result regards Family 1. In accordance with the biological modelling considerations above, let us assume hereafter that $j_2(T, p_T) = T$ and that the final proportion of mosquitoes in the populations is fixed $p(T) = p_T < 1$. The following result is a simplified and less precise version of Theorem 2.

Theorem A (Family 1)**.** *There exists $(T^*, u^*) \in \mathbb{R}_+ \times \mathcal{U}_{T,C,M}$ solving Problem $(\mathcal{P}^{1,\alpha}_{p_T,C,M})$. The overall behaviour of u^* depends on the convexity of $j_1(\cdot)$, the value of α and the value of C.*

In general, we distinguish the following cases:

- *Case 1. j_1 is either linear or strictly concave. There exists a real parameter $\alpha^* \in [0, 1)$ given by the parameters of the problem such that:*
  - *if C is large enough: If $\alpha \in [\alpha^*, 1]$, then $u^* = M\mathbb{1}_{[0,T^*]}$. If $\alpha \in (0, \alpha^*)$, then u^* is bang-bang with exactly one switch from M to 0 at a time $t_s \in (0, T^*)$ determined by α.*
  - *else, one has $u^* = M\mathbb{1}_{[0,C/M]}$.*

  In this case, the optimal time T^* reads

$$T^* = \int_0^{p_T} \frac{d\nu}{f(\nu) + u^*_p(\nu)g(\nu)} \quad \text{with } u^*_p(\nu) = M\mathbb{1}_{(0,p_s)} \text{ and } p_s = \begin{cases} p(t_s) & \text{if } C \text{ is large enough,} \\ p(C/M) & \text{otherwise.} \end{cases}$$

- *Case 2. j_1 is convex. If $\alpha \in (0, 1)$ singular controls may appear. The control u^* is non-decreasing until $t^* \in (0, T^*)$ such that $p(t^*) = p^*$ and then non-increasing.*

If $\alpha = 1$, the term with j_1 is no longer present and $u^* = M\mathbb{1}_{[0,\min\{T^*,C/M\}]}$.

Remark 1. We remark that in case j_1 is either linear or strictly concave the controls are always bang-bang (and the case $\alpha = 1$ is similar to $\alpha < 1$) while when j_1 is convex, singular controls may appear when $\alpha < 1$ while for $\alpha = 1$ the control is still bang-bang.

For our second result, regarding Family 2 let us assume hereafter that $j_2(T, p_T) = (1 - p_T)^2$ and that the time horizon $T > 0$ is fixed. The following result is a simplified and less precise version of Theorem 3.

Theorem B (Family 2)**.** *There exists $u^* \in \mathcal{U}_{T,C,M}$ solving Problem $(\mathcal{P}^{2,\alpha}_{T,C,M})$. In addition, there exists an interval (t^-, t^+) such that, outside of it $u^* = 0$ and the state p_{u^*} associated to u^* is constant. Inside (t^-, t^+), p_{u^*} is increasing and the behaviour of u^* depends on the convexity of $j_1(\cdot)$, the value of α and the values of C and T. We distinguish between the following cases:*

- *Case 1. j_1 is either linear or strictly concave. The solution is $u^* = M\mathbb{1}_{[t^-,t_s]}$, with $t_s \leqslant t^+$ the switching time.*

- *Case 2. j_1 is convex. If $\alpha \in (0,1)$ singular controls may appear. The control u^* is non-decreasing until $t^* \in (t^-, t^+)$ such that $p(t^*) = p^*$ and then non-increasing.*

If $\alpha = 1$, then $u^* = M\mathbb{1}_{[t^-,t_s]}$, with t_s defined as in the concave and linear case.

Remark 2. The use of bistable ODEs is widely spread to model a great variety of phenomena from biology, to physics and economy. Examples of systems with a bistable behaviour can be found in population dynamics, exploitation of natural resources, cell division, cancer modeling, apoptosis, chemical reactions or mechanical systems. Therefore, these results may be interesting outside of the particular context in which they have been presented.

In order to be able to apply Theorems A and B, $f, g \in \mathcal{C}^1([0,1])$ for the system considered must satisfy two conditions:

- **Bistability:** $p \mapsto f(p)$ must satisfy that $f(0) = f(1) = 0$, and that there exists a unique $\theta \in (0,1)$ such that $f(\theta) = 0$, $f(p) < 0$ for $p \in (0,\theta)$ and $f(p) > 0$ for $p \in (\theta, 1)$.

- **Increasingly costly to control:** $p \mapsto g(p)$ must be non-negative, strictly decreasing in $[0,1]$ and such that $g(1) = 0$. This means that as the state of the system gets closer to the steady state $p = 1$ it becomes increasingly harder to push.

Moreover, function $p \mapsto (f/g)(p)$ must satisfy two additional conditions, namely:

- **Unimodality**: $p \mapsto (f/g)(p)$ must be unimodal, that is, strictly decreasing for $p \in (0, p^*)$ and strictly increasing afterwards. With $0 < p^* < \theta$.

- $\lim_{p\to 1} (f/g)(p) = +\infty$.

1.4 Biological interpretation of our results and final comments

From a biological point of view, this problem is studied with more generality than what is strictly necessary. Only a certain subset of parameters is interesting for real field releases. In order to give a biological interpretation we restrict ourselves to the case where $p(T) > \theta$ so that the system in the long term tends to $p = 1$ without further action. Otherwise, once the releases ended the system would return to the initial condition after a certain time meaning that the installation of the Wolbachia-infected mosquito population would have failed. Independently of the family considered, with this restriction, our results yield:

- If j_1 is either linear or strictly concave, the optimal releasing strategy is bang-bang. Starting with $u^* = M$ and switching at most once, only after the critical proportion, $p(t) = \theta$, is surpassed.

- If j_1 is strictly convex, the possible appearance of singular solutions makes the analysis more intricate. In any case, solutions attain their maximal value at $t = t^*$ such that $p(t^*) = p^*$. Either u^* has a global maximum at t^* or there exists an open interval I where $u^*(t) = M$ and t^* belongs to I, although in the first case the value of the maximum attained at that point is not always straightforward to determine.

The function j_1 aggregates all the costs of the mosquito production, transport and release. Its convexity represents the marginal increase of the cost per mosquito. A concave function means that producing mosquitoes becomes proportionally less expensive as we scale up the production, while a convex function implies the opposite; the rate at which the costs increase grows as we increase the mosquito production. Finally, a linear j_1 means that the cost of production is scale-independent, directly proportional to the number of mosquitoes produced.

Since in a real case some of the parameters may be very difficult to determine beforehand, this interpretation gives us some guidelines to implement a sensible feedback strategy in the field. In order to do this, we would have to measure the proportion of infected mosquitoes using traps and adapt the amount of mosquitoes we release in consequence. We have shown that under a broad set of circumstances the best strategy is to act as soon as possible, and as fast as possible, at least until the critical value $p(t) = \theta$ is attained. An exception to this rule being the case when the production of mosquitoes is increasingly expensive. Nevertheless, in this context, the effort must also be concentrated soon, when the proportion of mosquitoes is $p(t) \approx p^*$, which allows to reduce the amount of mosquitoes used before reaching $p(t) = \theta$. This interpretation is, of course, extensible to any other problem fitting the hypothesis of Remark 2.

Methodological novelties and contributions. Beyond the particular application that motivated this work we would like to point out the contributions that this work does methodologically speaking. In this work, we prove a robustness property of the optimal control, with respect to the convexity/concavity of the increasing function j_1 defining the integrand term in the cost. More precisely, for this family of bistable systems (see Remark 2), controls are similar whether we consider a concave or a convex cost function. Let us highlight that to perform our analysis, we are led to introduce an appropriate change of variables using the monotonicity of the state variable, allowing us to significantly reduce the difficulty of the optimization problem, by transforming it into a simpler one of calculus of variations. Regarding the concave case, standard reasonings in calculus of variations cannot be applied directly, because of the lack of regularity of the functionals considered, which are not *a priori* lower semicontinuous. We introduce an alternative approach based on the investigation of optimality conditions for a finite dimensional auxiliary problem considered. This allows us to obtain both existence of an optimal control for the problem, but also a bang-bang property of minimizers (in other words, every minimizer is an extremal point of the set of admissible constraints).

2 Analysis of Family 1 problems

2.1 A first result: optimization without constraint on the number of mosquitoes used.

This section is devoted to studying the case where no constraint is imposed on the total number of mosquitoes used. In other words, we will deal with the optimal control problem

$$\begin{cases} \inf\limits_{\substack{u\in\mathcal{V}_{T,M}\\ T>0}} J_\alpha(T,u), \\ p' = f(p) + ug(p) \text{ in } (0,T),\ p(0)=0\ ,\ p(T)=p_T\ , \end{cases} \tag{$\mathcal{Q}^{1,\alpha}_{p_T,M}$}$$

where $J_\alpha(T,u)$ is defined by

$$J_\alpha(T,u) = (1-\alpha)\int_0^T j_1(u(t))dt + \alpha T, \tag{10}$$

where $\alpha \in [0,1]$, $j_1(\cdot)$ satisfies ($\mathcal{H}$) and $\mathcal{V}_{T,M}$ is given by

$$\mathcal{V}_{T,M} := \{u \in L^\infty(0,T), 0 \leqslant u \leqslant M \text{ a.e. in } (0,T)\}. \tag{11}$$

In what follows, it will be convenient to introduce the following notations:

$$m^*(p_T) := \max_{p\in[0,p_T]}\left(-\frac{f(p)}{g(p)}\right) > 0 \quad \text{and} \quad m_*(p_T) = \min_{p\in[0,p_T]}\left(-\frac{f(p)}{g(p)}\right) \leqslant 0. \tag{12}$$

for $p_T \in (0,1)$. Note that, as long as $p \mapsto (f/g)(p)$ satisfies the conditions of Remark 2, these quantities are unique.

Let us introduce the mapping F_0 defined by

$$v \mapsto F_0(v) := \frac{(1-\alpha)(vj_1'(v) - j_1(v)) - \alpha}{(1-\alpha)j_1'(v)}. \tag{13}$$

For the sake of notational simplicity, we do not underline the dependence of F with respect to α. A straightforward computation shows that F_0 is increasing (resp. decreasing) whenever j_1 is strictly convex (resp. strictly concave).

Theorem 1. *Let us assume that $\alpha \in (0,1]$, $p_T \in (0,1)$, (2) is true, and $j_1(\cdot)$ satisfies the first assumption of ($\mathcal{H}$). Let us assume that $M > m^*(p_T)$. Then, there exists a pair $(T^*,u^*) \in \mathbb{R}_+ \times \mathcal{V}_{T,M}$ solving Problem ($\mathcal{Q}^{1,\alpha}_{p_T,M}$). Moreover, let us distinguish between two cases:*

- The case where j_1 is either linear or strictly concave. *Let us introduce the real parameter $\alpha^* \in [0,1)$ given by*

$$\alpha^* = \frac{-m_* j_1(M)/M}{1 - m_* j_1(M)/M}. \tag{14}$$

In this case, if $\alpha \in [\alpha^*, 1]$, then $u^* = M\mathbb{1}_{[0,T^*]}$ and if $\alpha \in (0, \alpha^*)$, then u^* is bang-bang with exactly one switch from M to 0 at $t_s \in (0, T^*)$ such that

$$t_s = \int_0^{p_s} \frac{d\nu}{f(\nu) + Mg(\nu)} \quad \text{where } p_s \text{ is implicitly determined by } -\frac{f(p_s)}{g(p_s)} = \frac{-\alpha M}{(1-\alpha)j_1(M)}.$$

The optimal time T^* is given by

$$T^* = \int_0^{p_T} \frac{d\nu}{f(\nu) + u_p^*(\nu)g(\nu)} \quad \text{with } u_p^*(\nu) = M\mathbb{1}_{(0,p_s)},$$

with the convention that $p_s = p_T$ if $\alpha \in [\alpha^*, 1]$.

- *The case where j_1 is convex. In this case, define u_p^* as*

$$u_p^* : [0, p_T] \ni p_t \mapsto \max\{\min\{M, F_0^{-1}(-f(p_t)/g(p_t))\}, 0\}$$

If $\alpha \in (0,1)$ the optimal time T^* and control u^* read

$$T^* = \int_0^{p_T} \frac{d\nu}{f(\nu) + u_p^*(\nu)g(\nu)} \quad \text{and} \quad \forall t \in [0, T^*], \qquad u^*(t) = u_p^*(p_t)$$

where p_t denotes the unique solution in $[0, p_T]$ of the equation $t = \int_0^{p_t} \frac{d\nu}{f(\nu)+u_p^*(\nu)g(\nu)}$. If $\alpha = 1$ the same holds with $u_p^* = M\mathbb{1}_{[0,p_T]}$.

If $\alpha = 1$ the same holds with $u_p^* = M\mathbb{1}_{[0,p_T]}$.

Remark 3. A reasonable concern in the definition of $v \mapsto F_0(v)$ is its behavior in case $j_1'(0) = \infty$ or $j_1'(0) = 0$ like in the functions $u \mapsto j_1(u) := \sqrt{u}$ and $u \mapsto j_1(u) := u^2$. We can check, taking limits, that in these cases the reasoning is still valid and that the results obtained hold. The limit reads

$$\lim_{v\to 0} F_0(v) = \lim_{v\to 0} v - \frac{j_1(v)}{j_1'(v)} - \frac{\alpha}{(1-\alpha)j_1'(v)}$$

- If $j_1'(0) = \infty$, we obtain $\lim_{v\to 0} F_0(v) = 0$. In this case, $j_1(\cdot)$ must be concave and therefore $F_0(\cdot)$ decreasing, thus $F_0(v) < 0$ for all $v \in (0, M]$. Looking at the maximization conditions, (23), we see that this is consistent with the results.
- If $j_1'(0) = 0$ we can apply l'Hôpital's rule to find $\lim_{v\to 0} \frac{j_1(v)}{j_1'(v)} = \lim_{v\to 0} \frac{j_1'(v)}{j_1''(v)} = 0$ and therefore $\lim_{v\to 0} F_0(v) = -\infty$. This implies that we can never have $F(0) \geqslant 0$ and thus $u^* > 0$ for all $t \in (0, T^*)$.

For the sake of simplicity we showed this for $F_0(\cdot)$ but this remark will still be valid for the functions $F_\lambda(\cdot)$ we will introduce in 19.

Let us comment and illustrate the result above, by describing the behaviour of the solutions of Family 1, classified with respect to the convexity of $j_1(\cdot)$ and pointing out the limit values of α separating the different regimes.

Exploiting Theorem 2, we know that in the concave and linear cases, solutions are necessarily bang-bang. Either $u^* = M\mathbb{1}_{[0,T^*]}$ or with one switch from M to 0 occurring at time $t_s = \int_0^{p_s} \frac{d\nu}{f(\nu)+Mg(\nu)}$ with p_s solving $-\frac{f(p_s)}{g(p_s)} = \frac{-\alpha M}{(1-\alpha)j_1(M)}$. This happens if and only if $\alpha < \alpha^*$. The value of α separating both regimes is $\alpha^* = \frac{-m_* j_1(M)/M}{1-m_* j_1(M)/M}$. The existence and uniqueness of such a p_s is guaranteed under the hypothesis of Remark 2, and not exclusively for the f and g of this particular problem.

The convex case has a richer set of behaviours than the other ones. As an example, on Fig. 3 solutions are plotted for the particular choice $j_1(u) = e^{u/11} - 1$. This function is not intended to represent any realistic scenario but to illustrate the variety of possible solutions. The parameters considered for these simulations are given in Table 1, using the biological parameters considered in [3]. To obtain this plot, one needs to compute the function F_0^{-1} which has been done by using the nonlinear system solver of the software Python.

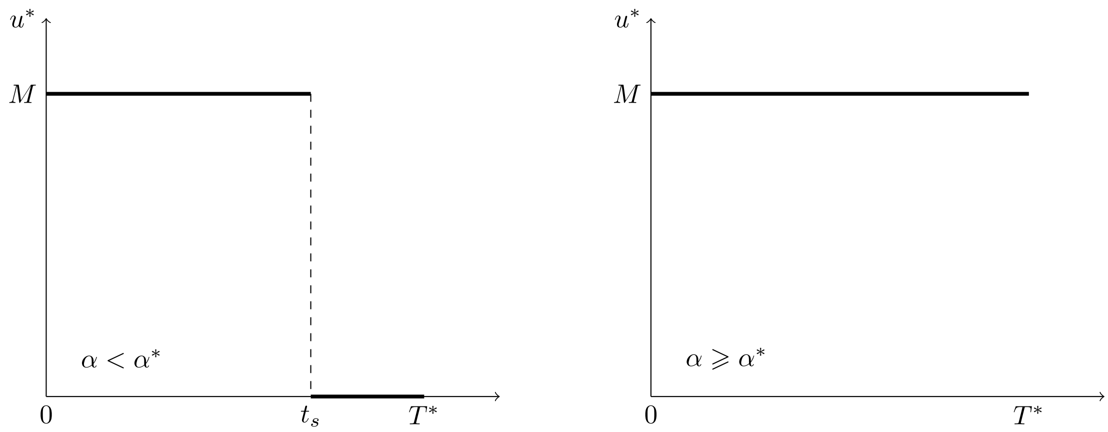

Figure 2: Control functions u^* solving problem ($\mathcal{Q}^{1,\alpha}_{p_T,M}$) in the linear and concave case.

Category	Parameter	Name	Value
Optimization	p_T	Final state	0.99
	M	Maximal instantaneous release rate	10
Biology	b_1^0	Normalized wild birth rate	1
	b_2^0	Normalized infected birth rate	0.9
	d_1	Wild death rate	0.27
	d_2	Infected death rate	0.3
	K	Normalized carrying capacity	1
	s_h	Cytoplasmatic incompatibility level	0.9

Table 1: Parameter values used to plot the solutions to problem ($\mathcal{P}^{1,\alpha}_{p_T,C,M}$)

The key factors to understand the behaviour of u^* in the convex case are the relative positions of $F_0(0)$ and $F_0(M)$ with respect to m_* and m^*. We begin by excluding the case $F_0(0) \geqslant m^*$ because for all $p_T \in (0,1)$, $F_0(0) \leqslant 0 < m^*$. Let us introduce

$$\alpha_0 \quad := \quad \frac{-m_* j_1'(0)}{1 - m_* j_1'(0)}, \tag{15}$$

$$\alpha_1 \quad := \quad \frac{M j_1'(M) - j_1(M) - m^* j_1'(M)}{1 + M j_1'(M) - j_1(M) - m^* j_1'(M)}, \tag{16}$$

$$\alpha_2 \quad := \quad \frac{M j_1'(M) - j_1(M) - m_* j_1'(M)}{1 + M j_1'(M) - j_1(M) - m_* j_1'(M)}. \tag{17}$$

These values are the thresholds separating the different regimes of the solutions. As an example, we deduce the value of α_1. If $M \geqslant F_0^{-1}(m^*)$ then $u_p^* : [0,p_T] \ni p_t \mapsto \max\{\min\{M, F_0^{-1}(-f/g(p_t))\}, 0\} = \max\{F_0^{-1}(-f/g(p_t)), 0\}$. Instead, if $M < F_0^{-1}(m^*)$, there will be an interval of positive measure in which $u_p^* = M$. Since F_0 depends on α, we can compute the smallest value of α for which the inequality $M \geqslant F_0^{-1}(m^*)$ holds:

$$F_0(M) := \frac{(1-\alpha)(M j_1'(M) - j_1(M)) - \alpha}{(1-\alpha) j_1'(M)} \geqslant m^* \Leftrightarrow \alpha \geqslant \frac{M j_1'(M) - j_1(M) - m^* j_1'(M)}{1 + M j_1'(M) - j_1(M) - m^* j_1'(M)} := \alpha_1$$

Here we assumed $M j_1'(M) - j_1(M) - m^* j_1'(M) \geqslant 0$, otherwise one can check that it is impossible to have $F_0(M) \geqslant m^*$. Doing a similar reasoning, one can see that we have similar equivalencies between $F_0(0) \leqslant m_*$ and $\alpha \geqslant \alpha_0$ and between $F_0(M) \leqslant m_*$ and $\alpha \geqslant \alpha_2$.

We conclude that the behaviors of the solution with respect to α are the following:

- If $\alpha \geqslant \alpha_0$, then $u^* > 0$ for a.e. $t \in (0, T^*)$, whereas if $\alpha < \alpha_0$ then there is an interval at the end in which $u^* = 0$.
- If $\alpha \leqslant \alpha_1$, then $u^* < M$ for a.e. $t \in (0, T^*)$.

- If $\alpha_1 < \alpha < \alpha_2$ an interval in which $u^* = M$ appears.
- Finally if $\alpha \geqslant \alpha_2$, $u^* = M$ for a.e. $t \in (0, T^*)$.

We recall that the function $x \mapsto \frac{x}{1+x}$ maps $[0, \infty)$ into $[0, 1)$. This implies that $(\alpha_0, \alpha_2) \in [0, 1)^2$ and that if $Mj_1'(M) - j_1(M) - m^* j_1'(M) \geqslant 0$, $\alpha_1 \in [0, 1)$ too. For the purpose of the discussion, in case $Mj_1'(M) - j_1(M) - m^* j_1'(M) < 0$ we can consider that $\alpha \geqslant \alpha_1$ always. Finally, since $x \mapsto \frac{x}{1+x}$ is increasing one only needs to compare the numerators of the expressions of α_0, α_1 and α_2 in order to compare their values. Computing this we obtain that $\alpha_2 \geqslant \alpha_0$ and $\alpha_2 \geqslant \alpha_1$. Nevertheless, in a general setting the relative position between α_1 and α_0 is not fixed.

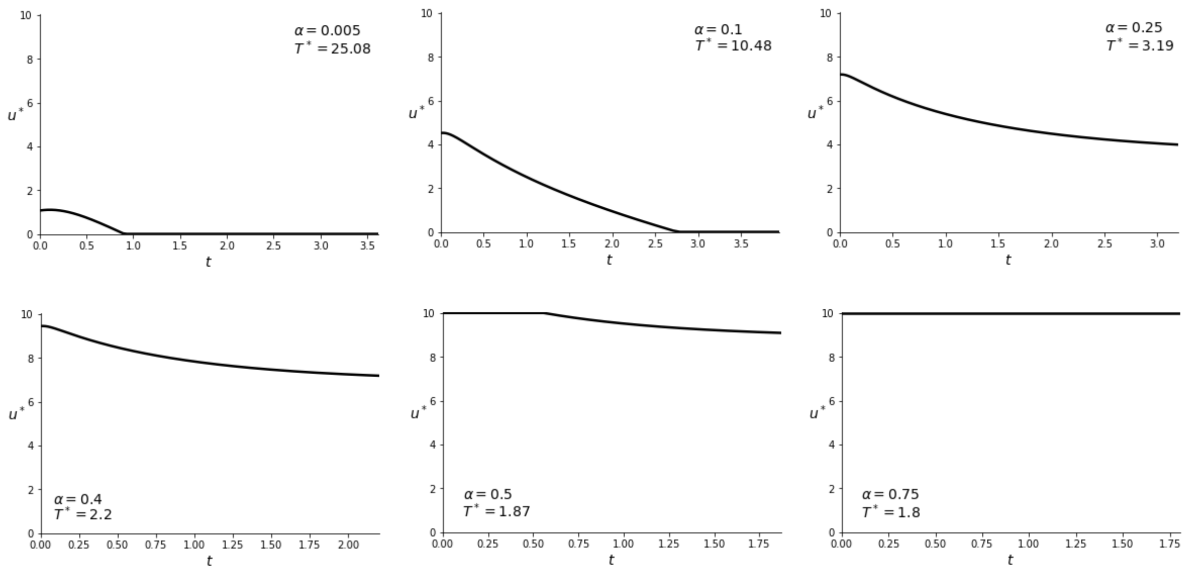

Figure 3: Control functions (T^*, u^*) solving problem $(\mathcal{P}^{1,\alpha}_{p_T,C,M})$ with $j_1(u) = e^{u/11} - 1$ as α increases, from left to right and from top to bottom. The values of α_0, α_1 and α_2 obtained are $\alpha_0 \approx 0.15$, $\alpha_1 \approx 0.44$ and $\alpha_2 \approx 0.55$. For the sake of clarity, for $\alpha = 0.005$ and $\alpha = 0.1$, u^* has not been represented in all its domain. Note that $u^* = 0$ in the rest of the domain.

2.2 Description of solutions

The following result characterizes the solutions to Family 1 problems.

Let us introduce

$$C^{p_T}(M) = \int_0^{p_T} \frac{M}{f(\nu) + Mg(\nu)} d\nu. \tag{18}$$

In this section we will assume, in accordance with the modeling issues discussed in Section 1.2, that $j_2(T, p_T) = T$. Therefore, (10) becomes

$$J_\alpha(T, u) = (1 - \alpha) \int_0^T j_1(u(t)) dt + \alpha T.$$

For $\alpha \in (0, 1)$, let us also introduce the mapping

$$v \mapsto F_\lambda(v) := \frac{(1 - \alpha)(v j_1'(v) - j_1(v)) - \alpha}{(1 - \alpha) j_1'(v) - \lambda}, \tag{19}$$

where $\lambda \in \mathbb{R}_-$ is a constant depending on the parameters of the problem, and the quantity

$$C_{\mathcal{Q}} := \int_0^{T^*_{\mathcal{Q}}} u^*_{\mathcal{Q}}(t) dt, \tag{20}$$

with $(T^*_{\mathcal{Q}}, u^*_{\mathcal{Q}})$ being the solution to the unconstrained case that has been treated in Theorem 1. Therefore, $C_{\mathcal{Q}}$ is the cost associated with this solution.

Nevertheless, we remark that existence properties for the optimal control problem ($\mathcal{P}^{1,\alpha}_{p_T,C,M}$), studied in Section A, are established in a more general setting, without prescribing explicitly the function j_2.

Theorem 2 (Family 1)**.** *Let us assume that $\alpha \in (0,1]$, $p_T \in (0,1)$, (2) is true, and $j_1(\cdot)$ satisfies the assumptions of ($\mathcal{H}$). Let us assume that $M > m^*(p_T)$ and*

$$C > C^{p_T}(M) \text{ if } p_T \leqslant \theta \qquad \text{and} \qquad C > C^{\theta}(M) \text{ otherwise.}$$

Then, there exists a pair $(T^*, u^*) \in \mathbb{R}_+ \times \mathcal{U}_{T,C,M}$ solving Problem ($\mathcal{P}^{1,\alpha}_{p_T,C,M}$). Moreover, let us distinguish two cases:

- *Case where j_1 is either linear or strictly concave. The optimal time and control are given by*

$$u^* = M\mathbb{1}_{[0,\min\{C_{\mathcal{Q}},C\}/M]} \quad \text{and} \quad T^* = \frac{\min\{C_{\mathcal{Q}},C\}}{M} + \int_{p_s}^{p_T} \frac{d\nu}{f(\nu)},$$

  with p_s solving $C^{p_s}(M) = \min\{C_{\mathcal{Q}}, C\}$.

- *Case where j_1 is convex. Let u^*_p be defined by*

$$u^*_p : [0,p_T] \ni p_t \mapsto \max\left\{\min\left\{M, F_\lambda^{-1}\left(-\frac{f(p_t)}{g(p_t)}\right)\right\}, 0\right\}$$

  If $\alpha \in (0,1)$ the optimal time T^* and control u^* are given by

$$T^* = \int_0^{p_T} \frac{d\nu}{f(\nu) + u^*_p(\nu)g(\nu)} \quad \text{and} \quad \forall t \in [0,T^*], \qquad u^*(t) = u^*_p(p_t)$$

  where p_t denotes the unique solution in $[0,p_T]$ of the equation $t = \int_0^{p_t} \frac{d\nu}{f(\nu)+u^*_p(\nu)g(\nu)}$ and λ is a Lagrange multiplier such that $\lambda = 0$ if, and only if, $C_{\mathcal{Q}} \leqslant C$.

If $\alpha = 1$ then $u^* = M\mathbb{1}_{[0,\min\{C^{p_T}(M),C\}/M]}$.

Remark 4. In case $C_{\mathcal{Q}} > C$, we have that $\lambda < 0$. Moreover, this value λ is implicitly determined by the equation $\int_0^{p_T} u^*_p(\nu)/(f(\nu) + u^*_p(\nu)g(\nu))d\nu = C$.

Remark 5. For $p_T \leqslant \theta$ and $C = C^{p_T}$ we still have existence of solutions, and indeed $u^* = M\mathbb{1}_{[0,T^*]}$ with $T^* = \int_0^{p_T} \frac{d\nu}{f(\nu)+Mg(\nu)} = \frac{C}{M}$. For the sake of clarity we exclude this case from the statement of the theorem, but it will be briefly discussed in the proof.

2.3 Proof of Theorem 1

The existence of solutions for Problem ($\mathcal{Q}^{1,\alpha}_{p_T,M}$) follows from an immediate adaptation of Proposition 3 and is left to the reader. Our approach is based on an adequate change of variable. In order to make this proof easier to follow, let us distinguish several steps.

Step 1: a change of variable for recasting the optimal control problem.

To introduce the adequate change of variable, we need the following result.

Lemma 1. *Let $(T^*, u^*) \in \mathbb{R}_+ \times \mathcal{V}_{T,M}$ solve Problem ($\mathcal{Q}^{1,\alpha}_{p_T,M}$) and let $\alpha > 0$. Let us introduce p_{u^*} solving*

$$\begin{cases} p'_{u^*} = f(p_{u^*}) + u^* g(p_{u^*}) & \text{in } (0,T^*) \\ p_{u^*}(0) = 0, \end{cases}$$

then one has $p'_{u^*}(t) > 0$ for all $t \in (0,T^*)$.

Proof. Let us argue by contradiction, assuming the existence of $0 \leqslant t_1 < t_2 \leqslant T$ such that $p_{u^*}(t_2) \leqslant p_{u^*}(t_1)$. Looking at the functional J_α we are minimizing, we claim that T^* is the smallest time at which $p_{u^*}(T^*) = p_T$. Indeed, since $\alpha > 0$, if there exists $T < T^*$ such that $p_{u^*}(T) = p_T$, the pair $(T, u^*|_{(0,T)})$ is admissible for Problem ($\mathcal{Q}^{1,\alpha}_{p_T,M}$), and moreover, $J_\alpha(T, u^*|_{(0,T)}) < J_\alpha(T^*, u^*)$ which contradicts the minimality of (T^*, u^*).

Let us first assume that $p_{u^*}(t_2) < p_{u^*}(t_1)$. Therefore, since $p_{u^*}(T) = p_T$, we infer by continuity the existence of $t_3 \in (t_2, T^*)$ such that $p_{u^*}(t_3) = p_{u^*}(t_1)$. Let us define $\tilde{u}$ as

$$\tilde{u}(t) = \begin{cases} u^*(t) & t \in (0, t_1), \\ u^*(t + t_3 - t_1) & t \in (t_1, \tilde{T}) \end{cases}$$

where $\tilde{T} = T^* - t_3 + t_1$. We proceed by direct comparison between the cost of both controls, obtaining

$$J_\alpha(T^*, u^*) - J_\alpha(\tilde{T}, \tilde{u}) \;\; = \;\; (1-\alpha)\int_{t_1}^{t_3} j_1(u^*(t))dt + \alpha(t_3 - t_1) \; > 0,$$

which contradicts the optimality of (T^*, u^*). The remaining case where $p_{u^*}(t_1) = p_{u^*}(t_2)$ can be treated similarly, by choosing $t_3 = t_2$. ☐

Let us now exploit this lemma in order to perform a useful change of variables that will allow us to reformulate Problem ($\mathcal{Q}^{1,\alpha}_{p_T,M}$). Given that $u \in \mathcal{V}_{T,M}$ solving Problem ($\mathcal{Q}^{1,\alpha}_{p_T,M}$) satisfies the necessary conditions $p(0) = 0$, $p(T) = p_T$ and $p'(t) = f(p(t)) + u(t)g(p(t)) > 0$ for all $t \in (0, T)$. Therefore, p defines a bijection from $(0, T)$ onto $(0, p_T)$. Denoting by $p^{-1} : [0, p_T] \to [0, T]$ its inverse, one has

$$p(t) = p_t \Leftrightarrow t = p^{-1}(p_t) = \int_0^{p_t} (p^{-1})'(\nu)d\nu = \int_0^{p_t} \frac{d\nu}{p'(p^{-1}(\nu))}$$

which leads to define the change of variable

$$t = \int_0^{p_t} \frac{d\nu}{f(\nu) + u(p^{-1}(\nu))g(\nu)}.$$

Introducing the function $p_t \mapsto u_p(p_t)$ defined by $u_p(p_t) := u(p^{-1}(p_t)) = u(t)$, one can easily infer that Problem ($\mathcal{Q}^{1,\alpha}_{p_T,M}$) is equivalent to

$$\inf_{u \in \hat{\mathcal{V}}_{p_T,M}} \hat{J}_{p,\alpha}(u_p), \tag{$\widehat{\mathcal{Q}}^{1,\alpha}_{p_T,M}$}$$

where $\hat{J}_{p,\alpha}(u_p)$ is defined by

$$\hat{J}_{p,\alpha}(u_p) = \int_0^{p_T} \frac{(1-\alpha)j_1(u_p(\nu)) + \alpha}{f(\nu) + u_p(\nu)g(\nu)} d\nu, \tag{21}$$

and $\hat{\mathcal{V}}_{p_T,M}$, is given by

$$\hat{\mathcal{V}}_{p_T,M} := \left\{u_p \in L^\infty(0, p_T), 0 \leqslant u_p \leqslant M \text{ a.e.}\right\}.$$

To recover the solution of ($\mathcal{Q}^{1,\alpha}_{p_T,M}$) from the solution of ($\widehat{\mathcal{Q}}^{1,\alpha}_{p_T,M}$), it suffices to undo the change of variable by setting $u(\cdot) = u_p(p(\cdot))$.

Note that, according to Lemma 1, the space $\hat{\mathcal{V}}_{p_T,M}$ is bigger than the space where solutions actually belong. The appropriate space is the range of $\mathcal{V}_{T,M}$, defined in (11), by the change of variable above, that is

$$\mathcal{W} := \left\{u_p \in L^\infty(0, p_T), f(p_t) + u_p(p_t)g(p_t) > 0 \text{ a.e.}\right\}.$$

It is notable that, as can be observed in Figure 4, one has

$$u_p \in \mathcal{W} \Leftrightarrow -f/g(\cdot) < u_p(\cdot) \leqslant M \text{ a.e. on } (0, \min\{p_T, \theta\}) \text{ and } 0 \leqslant u_p(\cdot) \leqslant M \text{ a.e. on } (\min\{p_T, \theta\}, p_T).$$

It follows from the definition of $\mathcal{W}$ that

$$\inf_{u \in \hat{\mathcal{V}}_{p_T,M}} \hat{J}_{p,\alpha}(u_p) \leqslant \inf_{u \in \mathcal{W}} \hat{J}_{p,\alpha}(u_p)$$

To solve the optimization problem in the right-hand side, we will solve Problem ($\widehat{\mathcal{Q}}^{1,\alpha}_{p_T,M}$), and check *a posteriori* that its solution $u_p^* \in \hat{\mathcal{V}}_{p_T,M}$ satisfies $u_p^* \in \mathcal{W}$ so that we will infer that

$$\inf_{u \in \mathcal{W}} \hat{J}_{p,\alpha}(u) = \inf_{u \in \hat{\mathcal{V}}_{p_T,M}} \hat{J}_{p,\alpha}(u) = \hat{J}_{p,\alpha}(u_p^*).$$

Step 2: first-order optimality conditions through the Pontryagin Maximum Principle.

Let us introduce, with a slight abuse of notation, the function t given by

$$t(p_t) = \int_0^{p_t} \frac{d\nu}{f(\nu) + u_p(\nu) g(\nu)}.$$

Let $U = [0, M]$. It is standard to derive optimality conditions for this problem[2] and one gets

$$u_p^*(p_t) \quad \in \quad \arg\max_{v \in U} - \frac{(1-\alpha) j_1(v) + \alpha}{f(p_t) + v g(p_t)}. \tag{22}$$

The case $\alpha = 1$ is obvious and leads to $u^*(\cdot) = M$ on $[0, T^*]$, after applying the inverse change of variable.

Let us now assume that $\alpha \in (0, 1)$. It is standard to introduce the switching function[3] ψ defined by

$$\psi(v) = -\frac{f(p_t)(1-\alpha) j_1'(v) + g(p_t)\left((1-\alpha)\left(v j_1'(v) - j_1(v)\right) - \alpha\right)}{\left(f(p_t) + v g(p_t)\right)^2},$$

and the maximization condition (22) yields

$$\begin{cases} \psi(0) \leqslant 0 \text{ on } \{u_p^* = 0\}, \\ \psi(u_p^*) = 0 \text{ on } \{0 < u_p^* < M\}, \\ \psi(M) \geqslant 0 \text{ on } \{u_p^* = M\}. \end{cases}$$

These functions allows us to write the aforementioned optimality conditions as

$$\begin{cases} F_0(0) \geqslant -\frac{f(p_t)}{g(p_t)} \text{ on } \{u_p^* = 0\}, \\ F_0(u_p^*) = -\frac{f(p_t)}{g(p_t)} \text{ on } \{0 < u_p^* < M\}, \\ F_0(M) \leqslant -\frac{f(p_t)}{g(p_t)} \text{ on } \{u_p^* = M\}. \end{cases} \tag{23}$$

where F_0 is given by (13).

Since the derivative of F_0 writes

$$F_0'(v) = (1-\alpha) j_1''(v) \frac{(1-\alpha) j_1(v) + \alpha}{\left((1-\alpha) j_1'(v)\right)^2},$$

this function shares the sign of $j_1''(v)$.

Step 3: analysis of the first-order optimality conditions.

Before discussing the different cases, it is useful to recall the behaviour of the function $p_t \mapsto -\frac{f(p_t)}{g(p_t)}$, represented in Figure 4. This function has two roots at $p_t = 0$ and $p_t = \theta$, is strictly positive between them and strictly negative after $p_t = \theta$, with a maximum at $p_t = p^*$ as defined in (5) and such that $\lim_{p_t \to 1} f(p_t)/g(p_t) = -\infty$ (See Remark 2). Another property that will be useful thereafter is that $p_t \mapsto f(p_t)/g(p_t)$ is not constant on any set of positive measure.

We conclude the proof looking each case separately:

- If $j_1''(\cdot) = 0$ then F_0 is constant, so that $F_0(0) = F_0(M)$ and u_p^* is necessarily *bang-bang*, equal to 0 or M a.e. in $(0, p_T)$ because $F_0(u_p^*) = -\frac{f(p_t)}{g(p_t)}$ cannot be constant. Looking at conditions (23) we see that if $F_0(0) \leqslant m_*$ the solution is $u_p^* = M\mathbb{1}_{[0,p_T]}$, since only the condition $F_0(M) \leqslant -\frac{f(p_t)}{g(p_t)}$ can be

[2] Indeed, one way consists in applying the Pontryagin Maximum Principle (PMP). Introducing the Hamiltonian $\mathcal{H}$ of the system, defined by

$$\begin{array}{rccl} \mathcal{H}: & (0,1) \times \mathbb{R}_+ \times \mathbb{R} \times \{0, -1\} \times U & \to & \mathbb{R} \\ & (p_t, t, \tau, q^0, u_p) & \mapsto & \frac{\tau + q^0((1-\alpha) j_1(u_p) + \alpha)}{f(p_t) + u_p g(p_t)}. \end{array}$$

where τ is the conjugated variable of t and satisfies $\tau' = -\partial_t \mathcal{H} = 0$ and therefore, τ is constant. Furthermore the transversality condition on τ yields $\tau = 0$. The instantaneous maximization condition reads $u_p^*(p_t) \in \arg\max_{v \in U} \mathcal{H}(p_t, t, \tau, q^0, v)$. Finally, since (τ, q^0) is nontrivial, one has $q^0 = -1$.

[3] Indeed, according to the PMP, the switching function is given by $\psi := \partial u_p \mathcal{H}(p_t, t, \tau, v)$.

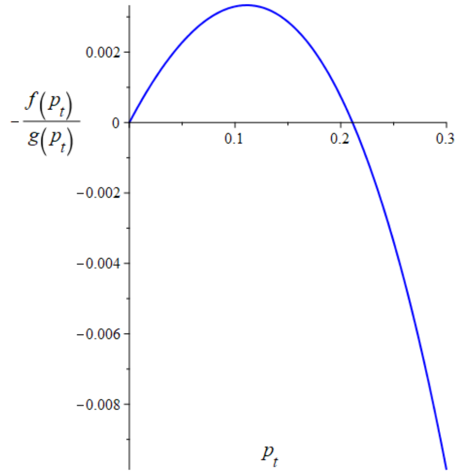

Figure 4: Function $p_t \mapsto -\frac{f(p_t)}{g(p_t)}$ represented between $p_t = 0$ and $p_t = 0.3$ for the parameters at table 1.

satisfied. On the other hand, if $F_0(0) > m_*$ then u_p^* has one switch from M to 0. We conclude by computing

$$F_0(0) = \frac{-\alpha}{(1-\alpha)j_1'(0)} \leqslant m_* \Longleftrightarrow \alpha \geqslant -m_* \frac{j_1'(0)}{1-m_* j_1'(0)} = \alpha^*$$

and noticing that $j_1'(0) = j_1(M)/M$.

- If $j_1''(\cdot) < 0$, then F is decreasing. We introduce the function Ψ we are maximizing, given by

$$\Psi(v) := -\frac{(1-\alpha)j_1(v) + \alpha}{f(p_t) + vg(p_t)}$$

and we recall that $\Psi' = \psi$. To show that u_p^* is *bang-bang*, let us use (22). For a given $p_t \in (0, p_T)$, let $N(v)$ be the numerator of $\psi(v)$. If there exists $v_0 \in (0, M)$ maximizing $\Psi(\cdot)$, then $f(p_t) + v_0 g(p_t) > 0$ according to Lemma 1. Moreover, $\psi(v_0) = N(v_0) = 0$ since v_0 is a critical point of Ψ and $\Psi''(v_0) = \psi'(v_0) \leqslant 0$. We compute

$$\psi'(v_0) = \frac{N'(v_0)\left(f(p_t) + vg(p_t)\right)^2 - N(v_0) 2 \left(f(p_t) + vg(p_t)\right) g(p_t)}{\left(f(p_t) + vg(p_t)\right)^4} = \frac{N'(v_0)}{\left(f(p_t) + v_0 g(p_t)\right)^2}.$$

which has the same sign as $N'(v_0)$, and

$$N'(v_0) = -(1-\alpha) j_1''(v_0) \left(f(p_t) + v_0 g(p_t)\right).$$

Therefore, one has $\Psi''(v_0) > 0$ leading to a contradiction with the maximality of v_0. It follows that the points $v_0 \in (0, M)$ satisfying $F_0(v_0) = -\frac{f(p_t)}{g(p_t)}$ cannot maximize Ψ, which shows that any solution is *bang-bang*.

A straightforward computation shows that

$$\Psi(M) - \Psi(0) = -\frac{(1-\alpha)j_1(M)f(p_t) - \alpha M g(p_t)}{f(p_t)(f(p_t) + Mg(p_t))}.$$

According to the optimality conditions (22), and because of the variations of $-f/g$, one sees that if u_p^* has a switching point, then it necessarily occurs strictly after θ since $F_0(0) < 0$.

Hence, from the expression of $\Psi(M) - \Psi(0)$, we get that any switching point p_s solves the equation

$$-\frac{f(p_s)}{g(p_s)} = \frac{-\alpha M}{(1-\alpha)j_1(M)}$$

and we can compute that the smallest value of α for which this equation has a solution is the one such that $m_* = \frac{-\alpha M}{(1-\alpha)j_1(M)}$ which allows us to recover α^*.

- If $j_1''(\cdot) > 0$, then $F_0(\cdot)$ is increasing, and the three conditions (23) are mutually exclusive and are thus both necessary and sufficient. The function $p_t \mapsto -\frac{f(p_t)}{g(p_t)}$ is increasing until p^*, defined in (5), and then decreasing (See the unimodality condition in Remark 2). Since F_0 defines a bijection, the optimality conditions (23) rewrite

$$\begin{cases} 0 \geqslant F_0^{-1}\left(-\frac{f(p_t)}{g(p_t)}\right) & \text{on } \{u_p^* = 0\}, \\ u_p^* = F_0^{-1}\left(-\frac{f(p_t)}{g(p_t)}\right) & \text{on } \{0 < u_p^* < M\}, \\ M \leqslant F_0^{-1}\left(-\frac{f(p_t)}{g(p_t)}\right) & \text{on } \{u_p^* = M\}. \end{cases}$$

  The expected expression of u^* follows then easily.

In order to finish the proof, we have to check that the solution $u_p^* \in \hat{\mathcal{V}}_{p_T,M}$ belongs to $\mathcal{W}$. This two spaces only differ for $p_t \in [0,\theta)$. We have that $F_0(0) = \frac{-\alpha}{(1-\alpha)j_1'(0)} \leqslant 0$. This implies that $u_p^* \neq 0$ in $(0,\theta)$, because the optimality condition $0 \geqslant F_0^{-1}\left(-\frac{f(p_t)}{g(p_t)}\right)$ cannot be satisfied in any open interval inside $(0,\theta)$. In the concave and linear case, since the solution is bang-bang, this also means that $u_p^* = M$ in $(0,\theta)$, therefore $f(p_t) + u_p^*(p_t)g(p_t) > 0$ in $p_t \in (0,\theta)$. In the convex case, we need to prove that $f(p_t) + u_p^*(p_t)g(p_t) > 0$ also in case the solution is a singular control. In that case, u_p^* satisfies the equation $u_p^* = F_0^{-1}\left(-\frac{f(p_t)}{g(p_t)}\right)$. Now $F_0(\cdot)$ is increasing, and so is $F_0^{-1}(\cdot)$, therefore

$$F_0^{-1}\left(-\frac{f(p_t)}{g(p_t)}\right) > -\frac{f(p_t)}{g(p_t)} \Leftrightarrow -\frac{f(p_t)}{g(p_t)} > F_0\left(-\frac{f(p_t)}{g(p_t)}\right) \text{ for } p_t \in (0,\theta).$$

This is true if, and only if $v > F_0(v)$ for $v \in (0, m^*]$. We have that

$$v > F_0(v) \Leftrightarrow v > \frac{(1-\alpha)(vj_1'(v) - j_1(v)) - \alpha}{(1-\alpha)j_1'(v)} \Leftrightarrow 0 > -\frac{(1-\alpha)j_1(v) + \alpha}{(1-\alpha)j_1'(v)}.$$

All the terms in the last fraction are positive, yielding that $f(p_t) + u_p^*(p_t)g(p_t) > 0$ for all $p_t \in (0,\theta)$, which ends the proof.

2.4 Proof of Theorem 2

Let us first recall that existence of solutions for Problem $(\mathcal{P}^{1,\alpha}_{p_T,C,M})$ has been proved in Proposition 3.

Step 1: derivation of the first-order optimality conditions.

By mimicking the reasoning in the first step of the proof of Theorem 1, one shows that the conclusion of Lemma 1 still holds true in that case, in other words, the optimal state p_{u^*} is increasing in $[0,T^*]$. This allow us to reformulate Problem $(\mathcal{P}^{1,\alpha}_{p_T,C,M})$ by defining the change of variable

$$t : p_t \mapsto \int_0^{p_t} \frac{d\nu}{f(\nu) + u(p^{-1}(\nu))g(\nu)},$$

introducing the function $p_t \mapsto u_p(p_t)$ defined by $u_p(p_t) := u(p^{-1}(p_t)) = u(t)$, so that Problem $(\mathcal{P}^{1,\alpha}_{p_T,C,M})$ is equivalent to

$$\inf_{u \in \hat{\mathcal{U}}_{p_T,C,M}} \hat{J}_{p,\alpha}(u_p), \tag{$\hat{\mathcal{P}}^{1,\alpha}_{p_T,C,M}$}$$

where $\hat{J}_{p,\alpha}(u_p)$ is defined by (21) and $\hat{\mathcal{U}}_{p_T,C,M}$ is given by

$$\hat{\mathcal{U}}_{p_T,C,M} := \left\{ u_p \in L^\infty([0,p_T]), 0 \leqslant u_p \leqslant M \text{ a.e. }, \int_0^{p_T} \frac{u_p(\nu)}{f(\nu) + u_p(\nu)g(\nu)} d\nu \leqslant C \right\}.$$

To recover the solution of $(\mathcal{P}^{1,\alpha}_{p_T,C,M})$ from the solution of $(\hat{\mathcal{P}}^{1,\alpha}_{p_T,C,M})$, it suffices to undo the variable change by setting $u(\cdot) = u_p(p(\cdot))$. Note that, as pointed out in the step 1 of Section 2.3, we are solving the problem in $\hat{\mathcal{U}}_{p_T,C,M}$, a bigger space than the range of $\mathcal{U}_{T,C,M}$ by the change of variable introduced in Lemma 1. This range is $\hat{\mathcal{W}} = \mathcal{W} \cap \hat{\mathcal{U}}_{p_T,C,M}$. As we have seen before, solutions to Problems ($\hat{\mathcal{P}}^{1,\alpha}_{p_T,C,M}$) and

$$\inf_{u \in \hat{\mathcal{W}}} \hat{J}_{p,\alpha}(u_p), \tag{24}$$

coincide as long as the solutions to Problem ($\hat{\mathcal{P}}^{1,\alpha}_{p_T,C,M}$) satisfy $f(p_t) + u_p^*(p_t)g(p_t) > 0$. Mimicking the reasoning at the end Step 3 in Section 2.3, one can similarly check that solutions to both problems above still coincide.

Let us derive and analyze optimality conditions for this problem. To handle the constraint

$$\int_0^{p_T} \frac{u_p(\nu)}{f(\nu) + u_p(\nu)g(\nu)} d\nu \leqslant C,$$

we introduce the mapping

$$p_t \mapsto z_p(p_t) = \int_0^{p_t} \frac{u_p(\nu)}{f(\nu) + u_p(\nu)g(\nu)} d\nu.$$

By following the same lines as in the proof of Theorem 1 and applying the PMP, one gets the existence of $\lambda \leqslant 0$ such that

$$\lambda \leqslant 0 \ , \qquad \lambda(z_p(p_T) - C) = 0. \quad \text{(transversality and slackness condition)} \tag{25}$$

and the optimal control u_p^* solves

$$u_p^*(p_t) \quad \in \quad \arg\max_{v \in U} \frac{\lambda v + q^0((1-\alpha)j_1(v) + \alpha)}{f(p_t) + vg(p_t)}. \tag{26}$$

In what follows, if $p_T \leqslant \theta$, we will assume without loss of generality that

$$C > C^{p_T}(M), \tag{27}$$

the case $C = C^{p_T}(M)$ being straightforward (in that case, one has necessarily $u_p^*(p_t) = M$ on $[0, p_T]$).

Let us show that $q^0 = -1$. To this aim, let us assume by contradiction that $q^0 = 0$. Hence, the optimality condition reads $u_p^*(p_t) \in \arg\max_{v \in U} \psi(v)$ where $\psi(v) = \frac{\lambda v}{f(p_t)+vg(p_t)}$, and since the 3-tuple of Lagrange multipliers is nontrivial according to the PMP, we necessarily have $\lambda < 0$ which, by condition (25), implies in turn that $z_p(p_T) = C$. If $p_t \in (0, \theta)$ (resp. $p_t \in (\theta, 1)$), ψ is increasing (resp. decreasing). Hence, if $p_T \leqslant \theta$, then $u_p^* = M\mathbb{1}_{[0,p_T]}$. This allows us to write $z_p(p_T) = Mt(p_T)$ leading to a contradiction since $C > Mt(p_T) = z_p(p_T)$ (See Remark 5). On the other hand, if $p_T > \theta$ the final state cannot be reached since $u_p^* = M\mathbb{1}_{[0,\theta]} + 0\mathbb{1}_{[\theta,p_T]}$. Given that with this control, p_{u^*} cannot attain p_T (remaining indefinitely at $p_{u^*} = \theta$) we reach again a contradiction. Therefore, it follows that $q^0 = -1$.

Step 2: analysis of the first-order optimality conditions.

Before discussing further the optimality conditions of this problem we remark a key fact in this proof. We introduce

$$C_{\mathcal{Q}} := \int_0^{T^*_{\mathcal{Q}}} u^*_{\mathcal{Q}}(t) dt,$$

where $\left(T^*_{\mathcal{Q}}, u^*_{\mathcal{Q}}\right)$ is the solution to Problem ($\mathcal{Q}^{1,\alpha}_{p_T,M}$) for the same value of α considered. Since $\mathcal{U}_{T,C,M} \subset \mathcal{V}_{T,M}$ we have that

$$\inf_{u_p \in \mathcal{V}_{T,M}} \hat{J}_{p,\alpha}(u_p) \leqslant \inf_{u_p \in \mathcal{U}_{T,C,M}} \hat{J}_{p,\alpha}(u_p).$$

This implies that if $C \geqslant C_{\mathcal{Q}}$, then $u^* = u^*_{\mathcal{Q}}$. Moreover, we can also deduce that, in case $C < C_{\mathcal{Q}}$ the constraint $z_p(p_T) \leqslant C$ is always saturated. By contradiction, if $z_p(p_T) < C$, then the slackness condition yields $\lambda = 0$. Therefore $u_p^*(p_t) \in \arg\max_{v \in U} -\frac{(1-\alpha)j_1(v)+\alpha}{f(p_t)+vg(p_t)}$, but this is the optimality condition for the unconstrained case and $u^*_{\mathcal{Q}} \notin \mathcal{U}_{T,C,M}$. Thus, the constraint must be saturated and we must have $\lambda < 0$. Consequently, we consider $C < C_{\mathcal{Q}}$ and $\lambda < 0$ from now on.

We begin by discussing the case $\alpha = 1$. From the optimality condition (26) we can derive

$$\begin{cases} \frac{1}{\lambda} \geqslant -\frac{f(p_t)}{g(p_t)} \text{ on } \{u_p^* = 0\}, \\ \frac{1}{\lambda} = -\frac{f(p_t)}{g(p_t)} \text{ on } \{0 < u_p^* < M\}, \\ \frac{1}{\lambda} \leqslant -\frac{f(p_t)}{g(p_t)} \text{ on } \{u_p^* = M\}. \end{cases}$$

From these conditions we see easily that u_p^* is *bang-bang*, since $p_t \mapsto \frac{f(p_t)}{g(p_t)}$ is not constant on any set of positive measure. Also, using the monotonicity of $p_t \mapsto \frac{f(p_t)}{g(p_t)}$ and the fact that $\lambda < 0$ we conclude that u_p^* has, at most, one switch from M to 0. Since the case without constraint had no switches and we are assuming $C < C_{\mathcal{Q}}$, it follows that $u_p^* = M\mathbb{1}_{[0,p_s]}$, with p_s solving $C^{p_s}(M) = C$. We can easily express this as a function of time since $C^{p_s}(M) = \int_0^{p_s} \frac{M}{f(\nu)+Mg(\nu)} d\nu = M \int_0^{p_s} \frac{d\nu}{f(\nu)+Mg(\nu)} = Mt_s$, thus $u^* = M\mathbb{1}_{[0,t_s]}$, with $t_s = \frac{C}{M}$.

Assuming now $\alpha \in (0,1)$ and following the same lines as in the unconstrained case we introduce

$$v \mapsto F_\lambda(v) := \frac{(1-\alpha)(vj_1'(v) - j_1(v)) - \alpha}{(1-\alpha)j_1'(v) - \lambda}.$$

Then, we can write the optimality conditions for Problem $(\hat{\mathcal{P}}^{1,\alpha}_{p_T,C,M})$ as

$$\begin{cases} F_\lambda(0) \geqslant -\frac{f(p_t)}{g(p_t)} \text{ on } \{u_p^* = 0\}, \\ F_\lambda(u_p^*) = -\frac{f(p_t)}{g(p_t)} \text{ on } \{0 < u_p^* < M\}, \\ F_\lambda(M) \leqslant -\frac{f(p_t)}{g(p_t)} \text{ on } \{u_p^* = M\}. \end{cases} \tag{28}$$

A straightforward computation shows that, like in the unconstrained case, $F_\lambda'(\cdot)$ and $j_1''(\cdot)$ have the same sign. This allows us to draw the same conclusions on the behaviour of u_p^* as in the unconstrained case. We sketch the reasoning hereafter:

- If $j_1''(\cdot) = 0$ then F_λ is constant and u_p^* *bang-bang*. Since $F_\lambda(0) = -\frac{\alpha}{(1-\alpha)j_1'(0)-\lambda} \leqslant 0$, then there is at most one switch. Moreover, we know that the constraint $z_p(p_T) \leqslant C$ is saturated and therefore that $u^* = M\mathbb{1}_{[0,t_s]}$ with $t_s = \frac{C}{M}$.
- If $j_1''(\cdot) < 0$, then F_λ is decreasing. *Mutatis mutandis*, we can reproduce the calculations done in Theorem 1, deducing that the behaviour of u_p^* is identical to the unconstrained case. That is, u_p^* is *bang-bang* with at most one switch from M to 0. Again, using the saturation of the constraint we deduce that $u^* = M\mathbb{1}_{[0,t_s]}$ with $t_s = \frac{C}{M}$.
- If $j_1''(\cdot) > 0$, then $F(\cdot)$ is increasing, and thus the three conditions (28) are both necessary and sufficient. This also implies that, once again, F_λ defines a bijection, so the optimality conditions (28) can be rewritten as

$$\begin{cases} 0 \geqslant F_\lambda^{-1}\left(-\frac{f(p_t)}{g(p_t)}\right) \text{ on } \{u_p^* = 0\}, \\ u_p^* = F_\lambda^{-1}\left(-\frac{f(p_t)}{g(p_t)}\right) \text{ on } \{0 < u_p^* < M\}, \\ M \leqslant F_\lambda^{-1}\left(-\frac{f(p_t)}{g(p_t)}\right) \text{ on } \{u_p^* = M\}. \end{cases}$$

  From these conditions we can do a straightforward derivation of the expression of u^*.

Remark 6. Note that the control u^* in the convex case with constraint has a very similar expression to the unconstrained case. Indeed the monotonicity of $p_t \mapsto F_\lambda^{-1}(-f/g(p_t))$ is the same: increasing until $p_t = p^*$ and then decreasing. This translates into u^* being non-decreasing until t^*, solving $p_{u^*}(t^*) = p^*$ and non-increasing afterwards. The relative positions of $F_\lambda(0)$ and $F_\lambda(M)$ with respect to m_* and m^* still play the same crucial role in the behaviour of solutions. Nevertheless, the values of α_0, α_1 and α_2 do not make sense anymore, since F_λ depends on λ which may change for different choices of α and C.

3 Analysis of Family 2 problems

3.1 Description of solutions

In this section we present and discuss the results obtained for the problem ($\mathcal{P}^{2,\alpha}_{T,C,M}$) of Family 2. As discussed in Section 1.2 let us assume $j_2(T,p(T)) = (1-p(T))^2$. Therefore, (10) becomes

$$J_\alpha(u) = (1-\alpha)\int_0^T j_1(u(t))dt + \alpha(1-p(T))^2.$$

In this family the time horizon T is fixed and $p(T)$ is free. The existence issues in a broader setting are treated separately in Appendix A.

We introduce the following notations in order to state the main result of this section:

$$M^* := \max_{p\in[0,1]}\left(-\frac{f'(p)}{g'(p)}\right) \tag{29}$$

Let us also introduce also p^{max} and $\bar{p}$ defined in the following way:

- If $C \leqslant C^\theta$,
$$p^{max} \quad \text{solves} \quad \int_0^{p^{max}} \frac{d\nu}{f(\nu)+Mg(\nu)} = \min\left\{\frac{C}{M}, T\right\}.$$
- If $C > C^\theta$,
$$p^{max} \quad \text{solves} \quad \int_0^{p^{max}} \frac{d\nu}{f(\nu)+M\mathbb{1}_{(0,\bar{p})}g(\nu)} = T.$$

where $\bar{p}$ is such that

$$\int_0^{\bar{p}} \frac{d\nu}{f(\nu)+Mg(\nu)} = \min\{C/M, T\}. \tag{30}$$

We remark that in the first case we have $p^{max} \leqslant \theta$.

Let us also introduce the mapping

$$v \mapsto F_{\lambda,\tau}(v) := \frac{vj_1'(v) - j_1(v) + \tau}{j_1'(v) - \lambda},$$

where $\lambda, \tau \in \mathbb{R}_-$.

Finally let us define

$$\alpha^0 := \frac{Kj_1(M)/M}{2 + Kj_1(M)/M} \quad \text{and} \quad \alpha^{max} = \frac{j_1(M)/\left(f(p^{max}) + Mg(p^{max})\right)}{2(1-p^{max}) + j_1(M)/\left(f(p^{max}) + Mg(p^{max})\right)}.$$

Note that both parameters satisfy $\alpha^0, \alpha^{max} \in (0,1)$ and, assuming $M \geqslant m^*(p_T)$ and $M > M^*$, they satisfy the inequality $\alpha^0 \leqslant \alpha^{max}$ [4]. Here, K denotes the environmental carrying capacity (see (1)). It appears in the definition of α^0 and hereafter due to the fact that $g(0) = 1/K$.

Theorem 3. *Let us assume that (2) is true, and that $j_1(\cdot)$ satisfies the assumptions of ($\mathcal{H}$). Let us assume that $M > m^*(p_T)$ and $\alpha \in (0,1]$[5]. Then, there exists a control $u^* \in \mathcal{U}_{T,C,M}$ solving problem ($\mathcal{P}^{2,\alpha}_{T,C,M}$) and times $t^-, t^+ \in [0,T]$ such that $u^* = 0$ in $(0,t^-) \cup (t^+,T)$ and in (t^-,t^+):*

- *Case where j_1 is either linear or strictly concave. The optimal control is $u^* = M\mathbb{1}_{[t^-,t_s]}$, with $t_s \leqslant t^+$. Assuming further that $M > M^*$ we have that*

[4] In order to prove this we recall that $x \mapsto \frac{x}{2+x}$ is an increasing function of x, we have $\alpha^0 \leqslant \alpha^{max}$ if and only if

$$K\frac{j_1(M)}{M} \leqslant \frac{j_1(M)}{(1-p^{max})(f(p^{max}) + Mg(p^{max}))}.$$

Reordering this we get $M \geqslant K(1-p^{max})(f(p^{max}) + Mg(p^{max}))$. Note that for $p^{max} = 0$ we have the equality, therefore we want to be sure that $p \mapsto (1-p)(f(p) + Mg(p))$ is non-increasing. Computing the derivative we obtain $-K(f(p) + Mg(p)) + K(1-p)(f'(p) + Mg'(p))$. The conditions needed for both terms to be individually smaller than zero are, precisely, $M > m^*(p_T)$ and $M > M^*$.

[5] We exclude the case $\alpha = 0$ for simplicity. Note that in that case the answer is trivially $u^* = 0$ a.e. in $[0,T]$.

– *If $\alpha \leqslant \alpha^0$, $u^* = 0$ for all $t \in (t^-, t^+)$.*

– *If $\alpha^0 < \alpha < \alpha^{max}$ then $u^* = M\mathbb{1}_{[t^-,t_s]}$ with t_s the smallest possible value such that $p_{u^*}(T) = p_T^*$, p_T^* being the only solution to $(1-p_T^*)\left(f(p_T^*) + Mg(p_T^*)\right) = \frac{1-\alpha}{2\alpha} j_1(M)$. This value can be explicitly computed: if $p_T^* \leqslant \theta$, then $t_s = T$ and if $p_T^* > \theta$ then t_s solves $t_s - t^- = \int_0^{p_s} \frac{d\nu}{f(\nu)+Mg(\nu)}$, with p_s the solution of $\int_0^{p_T^*} \frac{d\nu}{f(\nu)+M\mathbb{1}_{(0,p_s)}g(\nu)} = T$.*

– *If $\alpha \geqslant \alpha^{max}$ then $u^* = M\mathbb{1}_{[t^-,t_s]}$ with t_s solving $t_s - t^- = \int_0^{\bar{p}} \frac{d\nu}{f(\nu)+Mg(\nu)}$.*

- *Case where j_1 is convex. Let u_p^* be defined by*

$$u_p^* : [0, p_T] \ni p_t \mapsto \max\{\min\{M, F_{\lambda,\tau}^{-1}(-f/g(p_t))\}, 0\}$$

If $\alpha \in (0,1)$ the optimal control u^* reads $u^*(t) = u_p^*(p_t)$ for all $t \in [t^-, t^+]$, where p_t denotes the unique solution in $[0, p_T]$ to the equation $t = \int_0^{p_t} \frac{d\nu}{f(\nu)+u_p^*(\nu)g(\nu)}$.

If $\alpha = 1$ then $u^* = M\mathbb{1}_{[t^-,t^+]}$.
Moreover, calling $T^* \equiv t^+ - t^-$:

- *If $p_{u^*}(T) < \theta$, then $t^+ = T$*
- *If $p_{u^*}(T) = \theta$, control functions u_ξ^* such that $u_\xi^*(\cdot) = u^*(\cdot - \xi)$ a.e. with $\xi \in [-t^-, T - t^+]$ are also solutions.*
- *If $p_{u^*}(T) > \theta$, then $(t^-, t^+) = (0, T)$, thus $T^* = T$.*

Remark 7. Analogously to Family 1, in the convex case, λ and τ are equal to zero in case the constraints $\int_0^{p(T)} u_p^*(\nu)/(f(\nu) + u_p^*(\nu)g(\nu))d\nu \leqslant C$ and $T^* \leqslant T$, respectively, are not saturated. If the constraints are saturated, λ and τ are defined implicitly by these equalities.

3.2 A first result: optimization with T free but bounded and p_T fixed

We begin by stating and proving an intermediate result that will be useful for proving Theorem 3. In this section we investigate a seemingly unrelated problem, where only the cost term is considered, the final state is fixed, and the final time is free, but bounded. With a slight abuse of notation let us introduce

$$\begin{cases} \inf\limits_{\substack{u \in \mathcal{V}_{T^*,M} \\ T^* \leqslant T}} J(T^*, u) \\ p' = f(p) + ug(p) \text{ in } (0, T^*),\ p(0) = 0\ ,\ p(T^*) = p_T, \end{cases} \tag{$\mathcal{Q}^{2,T}_{p_T,C,M}$}$$

where $J(T^*, u)$ is defined by

$$J(T^*, u) = \int_0^{T^*} j_1(u(t))dt. \tag{31}$$

For $\tau \in \mathbb{R}_-$, let us introduce the mapping

$$v \mapsto F_\tau(v) := \frac{v j_1'(v) - j_1(v) + \tau}{j_1'(v)}. \tag{32}$$

Theorem 4. *Let us assume that $p_T \in (0,1)$, that (2) is true, and that $j_1(\cdot)$ satisfies the assumptions of ($\mathcal{H}$). Let us assume that $M > m^*(p_T)$ and that*

$$T \geqslant \int_0^{p_T} \frac{d\nu}{f(\nu) + Mg(\nu)}$$

Then, there exists a pair $(T^*, u^*) \in [0, T] \times \mathcal{U}_{T,C,M}$ solving Problem ($\mathcal{Q}^{2,T}_{p_T,C,M}$). Moreover, let us distinguish between two cases:

- *Case where j_1 is either linear or strictly concave. The optimal time and control read*

$$u^* = M\mathbb{1}_{[0,t_s]} \quad \textit{and} \quad T^* = \int_0^{p_T} \frac{d\nu}{f(\nu) + M\mathbb{1}_{(0,p_s)}g(\nu)}.$$

Where p_s is the only solution to $t_s = \int_0^{p_s} \frac{d\nu}{f(\nu)+Mg(\nu)}$. Moreover, if $p_T \leqslant \theta$ then $t_s = T^*$ and if $p_T > \theta$ then t_s is such that $T^* = T$.

- *Case where j_1 is convex. Let u_p^* be defined by*

$$u_p^* : [0, p_T] \ni p_t \mapsto \max\{\min\{M, F_\tau^{-1}(-f/g(p_t))\}, 0\}.$$

The optimal time T^* and control u^* read

$$T^* = \int_0^{p_T} \frac{d\nu}{f(\nu) + u_p^*(\nu)g(\nu)} \quad \textit{and} \quad \forall t \in [0, T^*], \qquad u^*(t) = u_p^*(p_t)$$

where p_t denotes the unique solution in $[0, p_T]$ to the equation $t = \int_0^{p_t} \frac{d\nu}{f(\nu)+u_p^*(\nu)g(\nu)}$. Moreover, $\tau \in \mathbb{R}_-$ and if for $\tau = 0$, $T^* \leqslant T$ then $\tau = 0$, otherwise τ is implicitly determined by the equation $T^* = T$.

3.3 Proof of Theorem 4

In order to prove Theorem 4 we will follow similar steps to the ones in Family 1. The idea behind the proof is to recast Problem ($\mathcal{Q}^{2,T}_{p_T,C,M}$) into a problem of Family 1 with an extra constraint $T^* \leqslant T$. We find the desired results by performing a similar reasoning to the one carried out in the proof of Theorem 2. Recall that our conclusions hold true for a larger class of functions f and g (See Remark 2).

Step 1: recasting into a Family 1 control problem with T^* bounded.

Adapting slightly the reasoning in Lemma 1, we see that the result is valid for Problem ($\mathcal{Q}^{2,T}_{p_T,C,M}$). We can therefore repeat the change of variable performed in Sections 2.3 and 2.4, that is

$$t : p_t \mapsto \int_0^{p_t} \frac{d\nu}{f(\nu) + u(p^{-1}(\nu))g(\nu)} \quad \text{and} \quad u_p(p_t) := u(p^{-1}(p_t)) = u(t).$$

Let us introduce a new problem.

$$\inf_{u \in \hat{\mathcal{V}}_{p_T^*,M}} \hat{J}_p(u_p), \tag{$\hat{\mathcal{Q}}^{2,T}_{p_T,C,M}$}$$

where $\hat{J}_p(u_p) := \int_0^{p_T} \frac{j_1(u_p(\nu))}{f(\nu)+u_p(\nu)g(\nu)} d\nu$ and $\hat{\mathcal{V}}_{p_T^*,M}$ is given by

$$\hat{\mathcal{V}}_{p_T^*,M} := \left\{ u_p \in \hat{\mathcal{V}}_{p_T,M}, \int_0^{p_T} \frac{d\nu}{f(\nu) + u_p(\nu)g(\nu)} \leqslant T \right\}.$$

From this new problem we will be able to recover the solutions of ($\mathcal{Q}^{2,T}_{p_T,C,M}$) by undoing the change of variable.

Similarly to the analysis of the problems of Family 1, we should impose the restriction $f(p(t)) + u^*(t)g(p(t)) > 0$ for $t \in [0, T]$ in the control space. Once again, we will not impose it in order to simplify the derivation of the solutions. Using analogous arguments to those exposed in Section 2.3, one can easily check that the solutions we obtain indeed belong to the range of $\mathcal{U}_{T,C,M}$ by the change of variable used.

Step 2: first-order optimality conditions through the Pontryagin Maximum Principle.

In addition to the notations used so far, we introduce, abusing of the notation $t(p_t) := \int_0^{p_t} \frac{d\nu}{f(\nu)+u_p(\nu)g(\nu)}$ in order to handle the constraint $T^* \leqslant T$. Applying the PMP we find:

$$\tau \leqslant 0\ , \qquad \tau(t(p_T) - T) = 0, \quad \text{(transversality and slackness condition)}$$

with τ being a constant. The optimal control u_p^* solves

$$u_p^*(p_t) \quad \in \quad \arg\max_{v\in U} \frac{\tau + q^0 j_1(v)}{f(p_t)+vg(p_t)}. \tag{33}$$

We can check that, if $T > \int_0^{p_T} \frac{d\nu}{f(\nu)+Mg(\nu)}$, then $q^0=-1$. By the PMP the pair (τ, q^0) is non-trivial. Assuming $q^0=0$, this implies that $\tau<0$ and $u_p^* \equiv M\mathbb{1}_{[0,p_T]}$. Since $\tau<0$ by the slackness condition $T^*=T$ and $T^*=\int_0^{p_T}\frac{d\nu}{f(\nu)+Mg(\nu)}$. So, without loss of generality, for the rest of the proof we consider $T>\int_0^{p_T}\frac{d\nu}{f(\nu)+Mg(\nu)}$ and $q^0=-1$.

Step 3: analysis of the first-order optimality conditions.

In the same spirit as in Theorem 2, we introduce the switching function

$$\begin{aligned} v\mapsto\psi(v) \quad &:= \quad \frac{\partial\mathcal{H}}{\partial v}(p_t,t,\tau,v) \\ &= \quad \frac{-f(p_t)j_1'(v)-g(p_t)\left(vj_1'(v)-j_1(v)+\tau\right)}{\left(f(p_t)+vg(p_t)\right)^2}. \end{aligned}$$

The maximization condition yields

$$\begin{cases} \psi(0)\leqslant 0 \text{ on } \{u_p^*=0\}, \\ \psi(u_p^*)=0 \text{ on } \{0<u_p^*<M\}, \\ \psi(M)\geqslant 0 \text{ on } \{u_p^*=M\}. \end{cases}$$

Using the mapping $F_\tau(\cdot)$ introduced in (32) we can write the optimality conditions as

$$\begin{cases} F_\tau(0)\geqslant -\frac{f(p_t)}{g(p_t)} \text{ on } \{u_p^*=0\}, \\ F_\tau(u_p^*)= -\frac{f(p_t)}{g(p_t)} \text{ on } \{0<u_p^*<M\}, \\ F_\tau(M)\leqslant -\frac{f(p_t)}{g(p_t)} \text{ on } \{u_p^*=M\}. \end{cases} \tag{34}$$

We compute the derivative of F_τ

$$F_\tau'(v)=j_1''(v)\frac{j_1(v)-\tau}{j_1'(v)^2}.$$

The sign of $F_\tau'(\cdot)$ depends exclusively on the sign of $j_1''(\cdot)$, hence we can extract similar conclusions on the behaviour of u_p^* to the ones obtained in Theorem 2, namely, u_p^* is *bang-bang* in the linear case and the three optimality conditions are mutually exclusive in the convex case. As for the concave case, we can prove that u_p^* is *bang-bang* too. To do this it suffices to reproduce the computations carried out in Theorem 1 but with the switching function of this section. These results lead us to conclude that:

- If $j_1''(\cdot)\leqslant 0$, then $u^*=M\mathbb{1}_{[0,t_s]}$. Using Lemma 1 we obtain that if $p_T\leqslant\theta$, then $t_s=T^*=\int_0^{p_T}\frac{d\nu}{f(\nu)+Mg(\nu)}$, since there cannot be any switch. If $p_T>\theta$, since $\int_0^t j_1(M)ds$ is an increasing function of time, by direct comparison we find that the switching time must be as small as possible. Since $t_s=\int_0^{p_s}\frac{d\nu}{f(\nu)+Mg(\nu)}$, a smaller t_s implies a smaller p_s. Taking into account that $T^*=\int_0^{p_T}\frac{d\nu}{f(\nu)+M\mathbb{1}_{(0,p_s)}g(\nu)}$ we conclude that minimising t_s is equivalent to maximising T^*. Therefore t_s is such that $T^*=T$.

- If $j_1''(\cdot)>0$, the three optimality conditions are mutually exclusive and therefore necessary and sufficient. Applying F_τ^{-1} to both sides of the inequalities in (34) we obtain the expression in the statement for u_p^*.

  We conclude arguing by contradiction. Let us call T_τ^* the T^* obtained for a particular value of τ. If $T_0^*\leqslant T$ then, for bigger values of T, the slackness condition implies $\tau=0$ and therefore $T^*=T_0^*$. The only way we can have $\tau<0$ is in case $T_0^*>T$, and in that case, using again the slackness condition we need $T_\tau^*=T$. Looking at the definition of T_τ^* and u_p^*, we conclude that τ must have a value such that $\int_0^{p_T}\frac{d\nu}{f(\nu)+u_p^*(\nu)g(\nu)}=T$.

3.4 Proof of Theorem 3

In order to prove Theorem 3 we will characterize an interval in which $p'_{u^*} > 0$. In this interval we will be able to adapt some of the results seen so far, specially those of Theorem 4. The solution outside of this interval will be null.

Step 1: recasting into a Family 1 control problem with T^* bounded.

Lemma 2. *Let $u^* \in \mathcal{U}_{T,C,M}$ be a control solving ($\mathcal{P}^{2,\alpha}_{T,C,M}$) and let $\alpha > 0$. Let us introduce p_{u^*} solving*

$$\begin{cases} p'_{u^*} = f(p_{u^*}) + u^* g(p_{u^*}) & \text{in } (0,T), \\ p_{u^*}(0) = 0. \end{cases}$$

Then, there exists one single interval $(t^-, t^+) \subseteq (0,T)$ in which $p'_{u^*} > 0$. Moreover, outside of this interval, $u^* = 0$ and $p'_{u^*} = 0$, implying that $p_{u^*}(0) = p_{u^*}(t^-) = 0$ and $p_{u^*}(t^+) = p_{u^*}(T)$.

Proof. The proof will be done by contradiction and it will follow the same lines as the one carried in Lemma 1. Assuming $p_{u^*}(T) > 0$ (if $p_{u^*}(T) = 0$ the solution is trivially $u^* = 0$), there necessarily exists a non-zero measure set in which $p'_{u^*} > 0$. We call $t^- = \inf\{t \in (0,T) \mid p'_{u^*}(t) > 0\}$ and $t^+ = \sup\{t \in (0,T) \mid p'_{u^*}(t) > 0\}$, therefore $\{t \in (0,T) \mid p'_{u^*}(t) > 0\} \subseteq (t^-, t^+)$. We assume that there exists an interval of non-zero measure $(t_1, t_2) \subset (t^-, t^+)$ such that $p'_{u^*} \leqslant 0$ a.e. on (t_1, t_2).

We split the proof in two parts: first we assume $p_{u^*}(T) \leqslant \theta$, and we define $\tilde{u}$ as

$$\tilde{u}(t) = \begin{cases} 0 & t \in (0, t_2 - t_1) \\ u^*(t - t_2 + t_1) & t \in (t_2 - t_1, t_2), \\ u^*(t) & t \in (t_2, T). \end{cases}$$

We proceed by direct comparison between the cost of both controls, obtaining

$$\begin{aligned} J_\alpha(u^*) - J_\alpha(\tilde{u}) &= (1-\alpha)\left(\int_0^T j_1(u^*(t))dt - \int_0^T j_1(\tilde{u}(t))dt\right) + \alpha((1-p_{u^*}(T))^2 - (1-p_{\tilde{u}}(T))^2) \\ &= (1-\alpha)\int_{t_1}^{t_2} j_1(u^*(t))dt + \alpha((1-p_{u^*}(T))^2 - (1-p_{\tilde{u}}(T))^2). \end{aligned}$$

Since $p'_{\tilde{u}} = 0$ on $(0, t_2 - t_1)$ but $p'_{u^*} \leqslant 0$ in (t_1, t_2) and they are equal on intervals of the same length, it follows that $p_{\tilde{u}}(T) \geqslant p_{u^*}(T)$. Therefore $J_\alpha(u^*) - J_\alpha(\tilde{u}) \geqslant 0$ which leads to a contradiction if the inequality is strict. In order to have the equality we need $p'_{u^*} = 0$ in (t_1, t_2) and since we assumed $p_{u^*}(T) \leqslant \theta$ this can only happen if $p_{u^*}(t_1) = p_{u^*}(t_2) = \theta$ and $u^* = 0$ on (t_1, t_2). But in this case $t_2 = T$, $t_1 = t^+$, so $(t^-, t^+) = \{t \in (0,T) \mid p'_{u^*}(t) > 0\}$ anyway.

Next, we assume $p_{u^*}(T) > \theta$, and we define $\tilde{u}$ as

$$\tilde{u}(t) = \begin{cases} u^*(t) & t \in (0, t_1) \\ u^*(t + t_2 - t_1) & t \in (t_1, T - t_2 + t_1), \\ 0 & t \in (T - t_2 + t_1, T). \end{cases} \tag{35}$$

Comparing the cost of both controls we obtain again $J_\alpha(u^*) > J_\alpha(\tilde{u})$, because in this case $p_{\tilde{u}}(T) > p_{u^*}(T)$ always. This yields the desired contradiction.

Since $(t^-, t^+) = \{t \in (0,T) \mid p'_{u^*}(t) > 0\}$ we have that $u^* = 0$ and $p'_{u^*} = 0$ in $(0, t^-)$ and thus $p_{u^*}(t^-) = 0$. On the other hand, we have $p_{u^*}(t^+) \geqslant p_{u^*}(T)$. But we must also have $u^* = 0$ and $p'_{u^*} = 0$ in (t^+, T), otherwise at least one of the two terms in $J_\alpha(u^*)$ would be bigger, thus $p_{u^*}(t^+) = p_{u^*}(T)$. ☐

This lemma proves that $p_{u^*}(t)$ is a bijection from (t^-, t^+) onto $(p_{u^*}(t^-), p_{u^*}(t^+))$. A straightforward exploration of its consequences already proves the last part of Theorem 3. Since we must have $p' = 0$ in $(0, t^-) \cup (t^+, T)$ and $p_{u^*}(t^+) = p_{u^*}(T)$ it follows that:

- If $p_{u^*} < \theta$, then $t^+ = T$, otherwise we would have $p' < 0$ in (t^+, T) and $p_{u^*}(t^+) < p_{u^*}(T)$.

- If $p_{u^*} = \theta$, as long as the length of $(0,t^-)\cup(t^+,T)$ is the same, the length of each interval does not affect the functional $J_\alpha(u)$, hence the conclusion.
- If $p_{u^*}(T) > \theta$, we have $p_{u^*}(t^+) = p_{u^*}(T) > \theta$ so $t^+ = T$, otherwise we would have $p' > 0$ in (t^+, T) and $p_{u^*}(t^+) > p_{u^*}(T)$. We have also that $t^- = 0$,. By contradiction we can construct a function following the same principle as in (35) (setting $t_1 = 0$ and $t_2 = t^-$) and prove that u^* is not optimal.

Exploiting this lemma further we can repeat the change of variable of the previous theorems one more time, but only in the subinterval (t^-, t^+).

Let us introduce the following problem:

$$\inf_{\substack{u\in\mathcal{U}^T_{p_T,C,M}\\ p_T\in[0,1)}} \hat{J}_{p,\alpha}(p_T, u_p), \tag{$\hat{\mathcal{P}}^{2,\alpha}_{T,C,M}$}$$

where $\hat{J}_{p,\alpha}(p_T, u_p)$ is defined by

$$\hat{J}_{p,\alpha}(p_T, u_p) = (1-\alpha)\int_0^{p_T} \frac{j_1(u_p(\nu))}{f(\nu)+u_p(\nu)g(\nu)}d\nu + \alpha(1-p_T)^2, \tag{36}$$

$\alpha \in (0,1]$ and $\mathcal{U}^T_{p_T,C,M}$ is given by

$$\mathcal{U}^T_{p_T,C,M} := \left\{ u_p \in \hat{\mathcal{U}}_{p_T,C,M},\, \int_0^{p_T} \frac{d\nu}{f(\nu)+u_p(\nu)g(\nu)} \leqslant T \right\}.$$

We remark that thanks to the change of variable, actually $T^* = t^+ - t^- = \int_0^{p_T} \frac{d\nu}{f(\nu)+u_p(\nu)g(\nu)}$, therefore $\mathcal{U}^T_{p_T,C,M}$ can also be expressed as $\mathcal{U}^T_{p_T,C,M} := \left\{ u_p \in \hat{\mathcal{U}}_{p_T,C,M}, T^* \leqslant T \right\}$. To recover the solution of ($\mathcal{P}^{2,\alpha}_{T,C,M}$) on the interval (t^-, t^+) from the solution of ($\hat{\mathcal{P}}^{2,\alpha}_{T,C,M}$), we need to undo the change of variable by setting $u(\cdot) = u_p(p(\cdot))$. Next, we need to determine t^- and t^+ which will be done in the following steps. Finally, $u^* = 0$ in $(0, t^-)$ and in (t^+, T).

Similarly to the analysis of the problems of Family 1, according to Lemma 2, we should impose the restriction $f(p(t)) + u^*(t)g(p(t)) > 0$ for $t \in (t^-, t^+)$ in the control space. Once again, we will not impose it in order to simplify the derivation of the solutions. Using analogous arguments to those exposed before, one can check that the solutions we obtain indeed belong to the range of $\mathcal{U}_{T,C,M}$ by the change of variable introduced in Lemma 2.

Step 2: Finding p^{max} (case $\alpha = 1$).

Let us define

$$\Phi : [0,1) \ni p_T \mapsto \inf_{u_p\in\mathcal{U}^T_{p_T,C,M}} \int_0^{p_T} \frac{j_1(u_p(\nu))}{f(\nu)+u_p(\nu)g(\nu)}d\nu.$$

Thanks to Theorem 4 we know this problem has a solution for all $p_T \in [0,1)$ if T is big enough. And therefore we can rewrite Problem ($\hat{\mathcal{P}}^{2,\alpha}_{T,C,M}$) as a minimisation problem in one variable, namely:

$$\inf_{p_T\in[0,1)} (1-\alpha)\Phi(p_T) + \alpha\,(1-p_T)^2\,.$$

Nevertheless, in Theorem 4, no constraint on the total number of mosquitoes used was imposed. Moreover the final time T was supposed big enough for solutions to exist. In order to apply the results of Theorem 4 to prove Theorem 3 we need to establish first which values of p_T are reachable for a given set of constraints. In other words, depending on T, C and M, there will be values of p_T such that $\mathcal{U}^T_{p_T,C,M}$ is empty. We note this maximal value p^{max}. Once we have characterized the set $[0, p^{max}]$, inside it we can disregard the constraint on C and apply Theorem 4 to find the solution.

In order to find the value of p^{max} we study the case $\alpha = 1$ in Problem ($\mathcal{P}^{2,\alpha}_{T,C,M}$). We recall it

$$\inf_{u\in\mathcal{U}_{T,C,M}} (1-p(T))^2.$$

Indeed, when α is set to 1 we are maximising p_T for a given set of constraints C and T regardless of $j_1(\cdot)$. This problem, in the case $T > C/M$, is discussed and solved in [3]. There, it is proven that solutions are *bang-bang* and such that saturate the constraint $\int_0^T u^*(t)dt = C$. Combining this result with Lemma 2 and since we are only looking at the subinterval (t^-, t^+) where $p'_{u^*} > 0$, we conclude that solutions have at most one switch from M to 0, and only if $p_{u^*}(T) > \theta$. A straightforward extension of their results yields that in the more general case, where the $T > C/M$ is not imposed, we have that if $C \leqslant C^\theta$ then $p^{max} \leqslant \theta$ and solves

$$\int_0^{p^{max}} \frac{d\nu}{f(\nu) + Mg(\nu)} = \min\left\{\frac{C}{M}, T\right\}.$$

Instead if $C > C^\theta$, then $p^{max} > \theta$ and solves

$$\int_0^{p^{max}} \frac{d\nu}{f(\nu) + M\mathbb{1}_{(0,\bar{p})}g(\nu)} = T$$

where $\bar{p}$ is such that $\int_0^{\bar{p}} \frac{d\nu}{f(\nu)+Mg(\nu)} = \min\{C/M, T\}$.

Step 3: Finding p_T^*

Thanks to the previous step we can finally write the expression we want to minimize, that is

$$\inf_{p_T \in [0, p_T^{max}]} (1-\alpha)\Phi(p_T) + \alpha\,(1-p_T)^2.$$

Now, for all $p_T \in [0, p^{max}]$ we know that $\Phi(p_T)$ is well defined and that u_p^* solving Problem ($\hat{\mathcal{Q}}^{2,T}_{p_T,C,M}$) for a given p_T^* solving this minimization problem, will solve Problem ($\hat{\mathcal{P}}^{2,\alpha}_{T,C,M}$) too.

We write the optimality conditions

$$\begin{cases} (1-\alpha)K\frac{j_1(u_p^*(0))}{u_p^*(0)} - 2\alpha \geqslant 0 \quad \text{if} \quad p_T^* = 0 \\ (1-\alpha)\frac{j_1(u_p^*(p_T^*))}{f(p_T^*)+u_p^*(p_T^*)g(p_T^*)} - 2\alpha(1-p_T^*) = 0 \quad \text{if} \quad 0 < p_T^* < p^{max}, \\ (1-\alpha)\frac{j_1(u_p^*(p^{max}))}{f(p^{max})+u_p^*(p^{max})g(p^{max})} - 2\alpha(1-p^{max}) \leqslant 0 \quad \text{if} \quad p_T^* = p^{max}. \end{cases} \tag{37}$$

In the convex case, these necessary conditions are not enough to give an explicit answer in a general setting. The first condition not even being well defined since $u_p^*(0)$ can be arbitrarily close to 0. Nevertheless, we focus here in the concave and linear case where these conditions can be further exploited.

If $j_1''(\cdot) \leqslant 0$, using Theorem 4 we have $u^* = M\mathbb{1}_{[0,t_s]}$. The switching point happening only if $p_{u^*}(T) > \theta$. In case there is a switch, $u^*(p_T) = 0$ and therefore the only optimality condition that can be satisfied is $-2\alpha(1-p^{max}) \leqslant 0$. Therefore $p_T^* = p^{max}$ and $u_p^* = M\mathbb{1}_{[0,\bar{p}]}$.

In case $u^*(p_T^*) = M$, we can rewrite the optimality conditions (37) as

$$\begin{cases} (1-\alpha)K\frac{j_1(M)}{M} - 2\alpha \geqslant 0 \quad \text{if} \quad p_T^* = 0 \\ (1-\alpha)\frac{j_1(M)}{f(p_T^*)+Mg(p_T^*)} - 2\alpha(1-p_T^*) = 0 \quad \text{if} \quad 0 < p_T^* < p^{max}, \\ (1-\alpha)\frac{j_1(M)}{f(p^{max})+Mg(p^{max})} - 2\alpha(1-p^{max}) \leqslant 0 \quad \text{if} \quad p_T^* = p^{max}. \end{cases} \tag{38}$$

Assuming $M > M^*$, the three conditions are mutually exclusive. Let us show it by computing the derivative of the condition with respect to p_T and showing that it is strictly increasing

$$-(1-\alpha)\frac{j_1(M)}{(f(p_T)+Mg(p_T))^2}\left(f'(p_T)+Mg'(p_T)\right) + 2\alpha > 0,$$

which is equivalent to

$$f'(p_T) + Mg'(p_T) < \frac{2\alpha}{1-\alpha}\frac{(f(p_T)+Mg(p_T))^2}{j_1(M)}.$$

This inequality needs to be satisfied for all α, and the right hand side is non-negative and increasing in α, therefore we want to ensure $f'(p_T) + Mg'(p_T) < 0$. This is true for all p_T if and only if $M > \max_{p_T\in[0,1]}\left(-\frac{f'(p_T)}{g'(p_T)}\right) := M^*$[6]. We can distinguish three cases

[6]This requirement is not much stronger than the minimum required for the existence of solutions, $M > m^*(p_T)$. For instance, with the parameters considered in Table 1 we obtain $m^*(1) \approx 0.0033$ and $M^* \approx 0.077$. On the other hand, the value of M in this table has been fixed to be $M = 10$.

- If $\alpha \leqslant \alpha^0$ then $p_T^* = 0$. Therefore $u^* = 0$ for all $t \in [0, T^*]$.
- If $\alpha^0 < \alpha < \alpha^{max}$ then $0 < p_T^* < p^{max}$ and it is the only solution of the equation $(1 - p_T^*)(f(p_T^*) + Mg(p_T^*)) = \frac{1-\alpha}{2\alpha} j_1(M)$. If $p_T^* \leqslant \theta$ there will not be any switch. If $p_T^* > \theta$, then since the final state is fixed and $\int_0^t j_1(M)ds$ is an increasing function of time, the switching point will be the smallest possible such that $p_{u^*}(T) = p_T^*$, this is $u_p^* = M\mathbb{1}_{[0,p_s]}$ with p_s solving $T^* = \int_0^{p_T^*} \frac{d\nu}{f(\nu)+M\mathbb{1}_{(0,p_s)}g(\nu)} = T$.
- If $\alpha \geqslant \alpha^{max}$ then $p_T^* = p^{max}$. Therefore $u_p^* = M\mathbb{1}_{[0,\bar{p}]}$. In other words, the switch is only possible if the constraint on the total amount of mosquitoes is saturated.

Appendix

A Existence of solutions for Problems ($\mathcal{P}^{1,\alpha}_{p_T,C,M}$) and ($\mathcal{P}^{2,\alpha}_{T,C,M}$)

This section is devoted to studying existence issues for problems ($\mathcal{P}^{1,\alpha}_{p_T,C,M}$) and ($\mathcal{P}^{2,\alpha}_{T,C,M}$). Note that the existence property for Problem ($\mathcal{P}^{1,\alpha}_{p_T,C,M}$) is a bit more intricate to show since the horizon of time T is let free.

Nevertheless, we will have to distinguish between the case where j_1 is convex or concave: the first case is standard whereas the second one needs a particular approach.

The existence of solutions for problems ($\mathcal{P}^{1,\alpha}_{p_T,C,M}$) and ($\mathcal{P}^{2,\alpha}_{T,C,M}$) will be studied with less restrictive hypothesis on the regularity of $j_1(\cdot)$ and $j_2(\cdot)$. We introduce:

$$\left\{\begin{array}{l} j_1(\cdot) \text{ is a non-negative increasing function such that } j_1(0)=0, \\ \text{either strictly concave, linear or strictly convex on } (0,T). \\ \\ j_2(\cdot) \text{ is a non-negative function, strictly increasing w.r.t. its first variable} \\ \text{and strictly decreasing w.r.t. its second variable. Moreover, for all } p\in[0,1], \\ \displaystyle\lim_{T\to+\infty} j_2(T,p) = +\infty. \end{array}\right. \tag{$\mathcal{H}'$}$$

A.1 Existence for Problem ($\mathcal{P}^{2,\alpha}_{T,C,M}$) in the case where j_1 is convex

The proof is standard and rests upon the direct method in the calculus of variations.

Proposition 1. *Let us assume that $j_1(\cdot)$ and $j_2(\cdot)$ satisfy ($\mathcal{H}'$). Let $T>0$, $M>0$, $C>0$ and let us assume that j_1 is convex in $\mathbb{R}$ and that for every T, $p_T \mapsto j_2(T,p_T)$ is lower semi-continuous in $[0,1]$. Then, Problem ($\mathcal{P}^{2,\alpha}_{T,C,M}$) has a solution.*

Proof. Since $\mathcal{U}_{T,C,M}$ is non-empty, let us consider a minimizing sequence $(u_n)_{n\in\mathbb{N}} \in \mathcal{U}_{T,C,M}^{\mathbb{N}}$ for Problem ($\mathcal{P}^{2,\alpha}_{T,C,M}$). We have $0 \leqslant u_n \leqslant M$ a.e. in $(0,T)$ for all $n\in\mathbb{N}$ and, according to the Banach-Alaouglu theorem, we conclude that $\mathcal{U}_{T,C,M}$ is compact for the weak-star topology of $L^\infty(0,T)$. Therefore, up to a subsequence, u_n converges to u^* for the weak-star topology of $L^\infty(0,T)$, and by a property of the weak star convergence, one gets that $0 \leqslant u^* \leqslant M$ a.e. in $(0,T)$ and

$$\int_0^T u^*(t)\,dt = \lim_{n\to+\infty} \int_0^T u_n(t)\,dt = \lim_{n\to+\infty} \langle u_n, 1\rangle_{L^\infty,L^1} \leqslant C.$$

We thus infer that $u^* \in \mathcal{U}_{T,C,M}$.

Next, we consider $(p_n)_{n\in\mathbb{N}}$ where p_n solves $p_n' = f(p_n) + u_n g(p_n)$ in $(0,T)$ with $p_n(0)=0$. Using the fact that f and g are continuous in $[0,1]$ and since $0 \leqslant p_n \leqslant 1$ in $[0,T]$, we deduce that $(p_n')_{n\in\mathbb{N}}$ is bounded in $L^\infty(0,T)$. Hence, p_n is bounded in $W^{1,\infty}([0,T])$ and according to the Ascoli-Arzelá theorem, $(p_n)_{n\in\mathbb{N}}$ converges in $\mathcal{C}^0([0,T])$ to $p^* \in W^{1,\infty}(0,T)$ up to a subsequence. Now, let $\varphi \in H^1(0,T)$. One has

$$p_n(T)\varphi(T) - \int_0^T p_n(t)\varphi(t)\,dt = \int_0^T (f(p_n) + u_n g(p_n))\varphi$$

for all $n \in \mathbb{N}$. According to the previous considerations, extracting adequately subsequences and letting then n tend to $+\infty$ shows that

$$p^*(T)\varphi(T) - \int_0^T p^*(t)\varphi(t)\,dt = \int_0^T (f(p^*) + u^* g(p^*))\varphi.$$

Therefore, a standard variational analysis yields that p^* satisfies $p'^* = f(p^*) + u^* g(p^*)$ in $(0,T)$ with $p^*(0) = 0$.

Finally, in order to assure the existence of solutions, it remains to prove that

$$\lim_{n\to\infty} J_\alpha(u_n) \geqslant J_\alpha(u^*). \tag{39}$$

By convexity of j_1, the functional

$$\mathcal{U}_{T,C,M} \ni u \mapsto \int_0^T j_1(u(t))\,dt$$

is convex. Furthermore, it is easy to see that the functional $L^2(0,T;[0,M]) \ni u \mapsto \int_0^T j_1(u(t))\,dt$ is continuous for the strong convergence of $L^2(0,T;[0,M])$ (indeed, this follows from the fact that the strong convergence in L^2 implies pointwise one and from the dominated convergence theorem). Now, using that a convex function on a real locally convex space is lower semicontinuous if and only if it is weakly lower semicontinuous, we infer that

$$\liminf_{n\to\infty} \int_0^T j_1(u_n(t))\,dt \geqslant \int_0^T \liminf_{n\to\infty} j_1(u^*(t))\,dt.$$

Up to a subsequence, $(p_n(T))_{n\in\mathbb{N}}$ converges to $p^*(T)$ and it follows by assumption on j_2 that up to a subsequence, $\liminf_{n\to+\infty} j_2(T, p_n(T)) \geqslant j_2(T, p^*(T))$, whence (39). This concludes the proof. $\square$

A.2 Existence for Problem $(\mathcal{P}^{2,\alpha}_{T,C,M})$ in the case where j_1 is concave

The concave case is a bit more intricate than the convex one. Indeed, we strongly used the convexity of j_1 to prove the lower semicontinuity of the integral term in the definition of J_α. We overcome this difficulty by introducing an auxiliary problem where only bang-bang control functions with a finite number of switches are considered.

Proposition 2. *Let us assume that $j_1(\cdot)$ and $j_2(\cdot)$ satisfy $(\mathcal{H}')$. Let $\alpha \in (0,1]$, $T > 0$, $M > 0$, $C > 0$ and let us assume that j_1 is concave. Then, Problem $(\mathcal{P}^{2,\alpha}_{T,C,M})$ has a solution which is necessarily bang-bang, equal a.e. to 0 or M and with at most two switches.*

Proof. To deal with the concave case, we introduce the set

$$\mathcal{U}_N := \{u \in \mathcal{U}_{T,C,M}, u \text{ bang-bang equal a.e. to 0 or } M \text{ and having at most } N \text{ switches}\}.$$

Let $N \in \mathbb{N}^*$ be given and consider the auxiliary problem

$$\begin{cases} \inf_{u\in\mathcal{U}_N} J_\alpha(u) \\ p' = f(p) + ug(p) \quad ,\ p(0) = 0. \end{cases} \tag{$\mathcal{P}^N$}$$

We first claim that Problem $(\mathcal{P}^N)$ has a solution. Indeed, note first that $\mathcal{U}_N$ is compact for the strong topology of $L^1(0,T)$ (since a sequence of switching points converges up to a subsequence in $[0,T]$ according to the Bolzano-Weierstrass lemma). Let $(u_{N,n})_{n\in\mathbb{N}}$ denote a minimizing sequence for Problem $(\mathcal{P}^{2,\alpha}_{T,C,M})$. Up to a subsequence, $(u_{N,n})_{n\in\mathbb{N}}$ converges to some element u_N in $L^1(0,T)$. Since $j_1(\cdot)$ is locally Lipschitz as a concave function, there exists $K > 0$ such that

$$\left| \int_{(0,T)} j_1(u_{N,n}) - \int_{(0,T)} j_1(u_N) \right| \leqslant \int_0^T |j_1(u_{N,n}) - j_1(u_N)| \leqslant K \|u_{N,n} - u_N\|_{L^1(0,T)}.$$

Finally, dealing similarly as in the proof of Lemma 1 with the term $j_2(T, p_{N,n}(T))$, where $p_{N,n}$ stands for the solution to $p' = f(p) + u_{N,n} g(p)$ and $p(0) = 0$, enables us to show that (39) still holds true in that case. It follows that Problem $(\mathcal{P}^N)$ has a solution u_N.

Let us now show that u_N has at most two switches. Let $u_N \in \mathcal{U}_N$ solving Problem ($\mathcal{P}^N$). Let $0 \leqslant \xi_1 < ... < \xi_{N_0} \leqslant T$ denote the distinct switching points of u_N with $N_0 \leqslant N$, with the convention that $\xi_1 = 0$ if, and only if, $u_N = M$ in a neighborhood of $t = 0$ and that $x_{N_0} = T$ if, and only if, $u_N = M$ in a neighborhood of $t = T$. We have to distinguish between two cases: there exist three distinct switching points ξ_{k-1}, ξ_k and ξ_{k+1} such that (a) $\xi_{k-1} > 0$ and $u = M$ on (ξ_k, ξ_{k+1}), or (b) $\xi_{k+1} < T$ and $u = M$ on (ξ_{k-1}, ξ_k). In what follows, we will only deal with the case (a), the study of the case (b) being exactly similar.

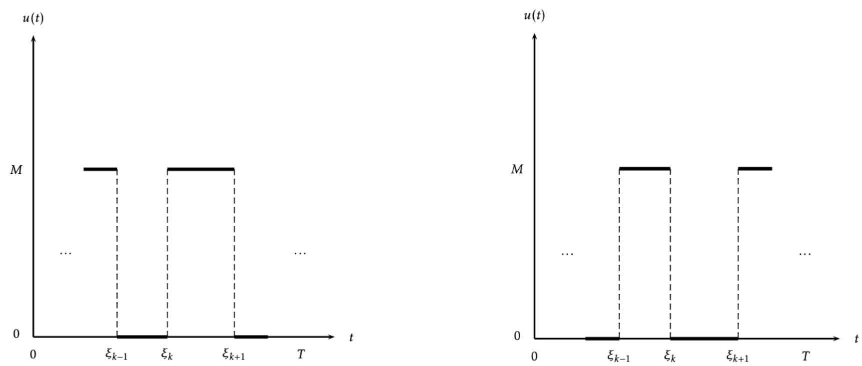

Figure 5: Left: case (a). Right: case (b).

Let us first write $J_\alpha(u_N)$ as a function of the ξ_k as

$$J_\alpha(u_N) := J_\alpha^\xi(\xi_1, \dots, \xi_{N_0}) := (1-\alpha) j_1(M) \sum_{\substack{j \in [\![1, N_0]\!] \\ j \text{ odd}}} (\xi_{j+1} - \xi_j) + \alpha j_2(T, p_N(T)),$$

where p_N denotes the solution to the Cauchy problem $p' = f(p) + u_N g(p)$ with $p(0) = 0$.

Hence, one can rewrite Problem ($\mathcal{P}^N$) as

$$\left\{ \begin{array}{l} \inf_{\xi_1 < \dots < \xi_{N_0}} J_\alpha^\xi(\xi_1, \dots, \xi_{N_0}), \\ \displaystyle\sum_{\substack{j \in [\![1, N_0]\!] \\ j \text{ odd}}} (\xi_j - \xi_{j-1}) \leqslant C/M, \\ \xi_j - \xi_{j+1} < 0 \ , \ j = 1, \dots, N_0 - 1 \end{array} \right.$$

Notice that this problem is equivalent to Problem ($\mathcal{P}^N$) and has therefore a solution. We write $p_N(T)$ in terms of the ξ_k as

$$p_N(T) = \sum_{\substack{j \in [\![1, N_0]\!] \\ j \text{ odd}}} \int_{\xi_j}^{\xi_{j+1}} \left(f(p_N(t)) + M g(p_N(t)) \right) dt.$$

Let k denote the integer satisfying the conditions of the case (a). Applying the Karush Kuhn-Tucker theorem to the optimization problem above in order to obtain the optimality conditions yields the existence of a Lagrange multiplier $\mu \in \mathbb{R}_+$ such that

$$\mu \left(\sum_{\substack{k \in [\![1, N_0]\!] \\ k \text{ odd}}} (\xi_k - \xi_{k-1}) - C/M \right) = 0 \qquad \text{(slackness condition)}$$

and

$$\left\{ \begin{array}{l} (-1)^k (1-\alpha) j_1(M) + (-1)^k \alpha \frac{\partial j_2}{\partial p}(T, p_N(T)) M g(p_N(\xi_k)) + (-1)^k \mu M = 0 \\ (-1)^{k+1} (1-\alpha) j_1(M) + (-1)^{k+1} \alpha \frac{\partial j_2}{\partial p}(T, p_N(T)) M g(p_N(\xi_{k+1})) + (-1)^{k+1} \mu M = 0. \end{array} \right.$$

Therefore, adding these two equations, we get

$$\alpha M \frac{\partial j_2}{\partial p}(T, p_N(T)) (g(p_N(\xi_k)) - g(p_N(\xi_{k+1}))) = 0.$$

Since $\alpha > 0$, $\frac{\partial j_2}{\partial p}(T, p_N(T)) < 0$ and $p \mapsto g(p)$ is strictly decreasing we reach a contradiction. It follows that u_N has at most two switches and we infer that

$$\inf_{u \in \mathcal{U}_N} J_\alpha(u) = \min_{v \in \mathcal{U}_2} J_\alpha(v) \tag{40}$$

To conclude, one needs to investigate the links between Problems ($\mathcal{P}^N$) and ($\mathcal{P}^{2,\alpha}_{T,C,M}$). One important ingredient is the following lemma, whose proof is postponed to the end of this section for the sake of readability.

Lemma 3. *Let u be an element of $\mathcal{U}_{T,C,M}$ such that u is bang-bang. Then, there exists u^N in $\mathcal{U}_N$ such that*

$$\lim_{N \to +\infty} u^N(t) = u(t) \quad \textit{for a.e. } t \in (0, T).$$

Let $u \in \mathcal{U}_{T,C,M}$. It is well-known that the set $\{v \in \mathcal{U}_{T,C,M} \mid v \text{ is bang-bang, equal a.e. to 0 or } M\}$ is dense into $\mathcal{U}_{T,C,M}$ for the weak-star topology topology of $L^\infty(0,T)$. Hence, there exists $(u_k)_{k \in \mathbb{N}} \in \mathcal{U}^{\mathbb{N}}_{T,C,M}$ converging weakly-star to u in $L^\infty(0,T)$.

By concavity of j_1 in $\mathbb{R}$, the mapping

$$\mathcal{U}_{T,C,M} \ni u \mapsto \int_0^T j_1(u(t))\, dt$$

is concave. Mimicking the argument used at the end of the proof of Proposition 1, one shows the upper semicontinuity property:

$$\limsup_{k \to +\infty} \int_{(0,T)} j_1(u_k) \leqslant \int_{(0,T)} j_1(u)$$

Now, according to Lemma 3, there exists $u_k^N \in \mathcal{U}_N$ such that

$$\lim_{N \to +\infty} u_k^N(t) = u_k(t) \quad \text{for a.e. } t \in (0, T).$$

The dominated convergence theorem thus yields

$$\lim_{N \to +\infty} \int_{(0,T)} j_1(u_k^N) = \int_{(0,T)} j_1(u_k).$$

According to the convergence results above, we infer that, $\varepsilon > 0$ being given, there exists $N_0 \in \mathbb{N}$ such that

$$N \geqslant N_0 \Longrightarrow \limsup_{k \to +\infty} \int_{(0,T)} j_1(u_k^N) - \varepsilon \leqslant \limsup_{k \to +\infty} \int_{(0,T)} j_1(u_k) \leqslant \int_{(0,T)} j_1(u).$$

Dealing with the term involving j_2 is easier. Indeed, by using the approximation results above and mimicking the reasoning in the proof of Proposition 1, one gets

$$\lim_{N,k \to +\infty} j_2(T, p_k^N(T)) = j_2(T, p(T))$$

where p_k^N (resp. p) denotes the solution to the Cauchy problem $p' = f(p) + u_k^N g(p)$ and $p(0) = 0$ (resp. $p' = f(p) + ug(p)$ and $p(0) = 0$). The combination of the convergence results above yields the existence of $\hat{N}_0 \in \mathbb{N}$ such that

$$N \geqslant \hat{N}_0 \Longrightarrow \limsup_{k \to +\infty} J_\alpha(u_k^N)) - \varepsilon \leqslant J_\alpha(u).$$

Since ε has been chosen arbitrarily, and since $J_\alpha(u_k^N) \geqslant \min_{v \in \mathcal{U}_2} J_\alpha(v)$ according to (40), one gets

$$J_\alpha(u) \geqslant \min_{v \in \mathcal{U}_2} J_\alpha(v).$$

This concludes the proof: Problem ($\mathcal{P}^{2,\alpha}_{T,C,M}$) has a solution which solves moreover Problem ($\mathcal{P}^2$). $\square$

Proof of Lemma 3. Since u is assumed to be bang-bang, let us write $u = M\mathbb{1}_I$ where I denotes a measurable subset of $(0,T)$. Let $\varepsilon > 0$. By outer regularity of the Lebesgue measure, there exists un open subset of $(0,T)$ containing I and such that $|I| \leqslant |O| \leqslant |I| + \varepsilon$. Let us write $O = \bigcup_{n\in\mathbb{N}}(\alpha_n, \beta_n)$ where the intervals (α_n, β_n) are disjoint and such that $|O| = \sum_{n\in\mathbb{N}}(\beta_n - \alpha_n)$. Let us introduce $u_n := M\mathbb{1}_{\bigcup_{p=0}^n(\alpha_p,\beta_p)}$. Writing $u = (u - M\mathbb{1}_O) + M\mathbb{1}_O$, one has

$$\begin{aligned}\int_0^T |u - u_n| &\leqslant M\int_0^T |\mathbb{1}_I - \mathbb{1}_O| + M\int_0^T |\mathbb{1}_O - \mathbb{1}_{\bigcup_{p=0}^n(\alpha_p,\beta_p)}| \\ &\leqslant 2\varepsilon M + M\sum_{p=n+1}^{+\infty}(\beta_p - \alpha_p)\end{aligned}$$

Since ε is arbitrary and since the series with general term $\beta_n - \alpha_n$ is convergent, it follows that $(u_n)_{n\in\mathbb{N}}$ converges to u in $L^1(0,T)$ and thus also pointwise. This concludes the proof. □

A.3 Existence results for Problem $(\mathcal{P}^{1,\alpha}_{p_T,C,M})$

Proposition 3. *Let us assume that $\alpha \in (0,1]$, $p_T \in (0,1)$, (2) is true, and that $j_1(\cdot)$ and $j_2(\cdot)$ satisfy the assumptions of $(\mathcal{H}')$. Let us assume that $M > m^*(p_T)$ and*

$$C > C^{p_T}(M) \text{ if } p_T \leqslant \theta \qquad \text{and} \qquad C > C^\theta(M) \text{ otherwise.}$$

Finally, let us also assume that for every p_T, $T \mapsto j_2(T, p_T)$ is lower semi-continuous in $\mathbb{R}_+$. Then, Problem $(\mathcal{P}^{1,\alpha}_{p_T,C,M})$ has a solution.

Proof. To avoid working on a variable domain, let us make the following change of variables: we define $\tilde p(s) := p(Ts)$ and $\tilde u(s) := u(Ts)$, with $s \in [0,1]$. Then, Problem $(\mathcal{P}^{1,\alpha}_{p_T,C,M})$ rewrites

$$\begin{cases} \inf\limits_{(T,\tilde u)\in\mathcal{D}^{p_T}} \tilde J_\alpha(T,\tilde u), \\ \tilde p'(s) = T\left(f(\tilde p(s)) + \tilde u(s)g(\tilde p(s))\right)\ ,\ \tilde p(0) = 0\ ,\ \tilde p(1) = p_T, \end{cases} \tag{$\tilde{\mathcal{P}}^{1,\alpha}_{p_T,C,M}$}$$

where $\tilde J(T,\tilde u)$ is defined by

$$\tilde J(T,\tilde u) = (1-\alpha)\ T\int_0^1 j_1(\tilde u(s))ds + \alpha j_2(T, p_T).$$

and $\mathcal{D}^{p_T}$ is the set of admissible controls

$$\mathcal{D}^{p_T} = \{(T,\tilde u) \in \mathbb{R}_+ \times \mathcal{U}_{1,C,M} \times [0,1] \mid \tilde p(1) = p_T\}.$$

Let us first prove that $\mathcal{D}^{p_T}$ is non-empty. To this aim, let us define

$$T^{p_T}(M) := \int_0^{p_T} \frac{d\nu}{f(\nu) + Mg(\nu)} \tag{41}$$

and look for controls of the form $u_\xi(t) = M\mathbb{1}_{[0,\xi]}$ belonging to this set, where

$$\xi = \begin{cases} 1 & \text{if } MT^{p_T} \leqslant C \\ \frac{C}{MT^{p_T}} & \text{if } MT^{p_T} > C. \end{cases} \tag{42}$$

Let us introduce $\tilde p_\xi$ solving $\tilde p_\xi' = T\left(f(\tilde p_\xi) + \tilde u_\xi g(\tilde p_\xi)\right)$ in $(0,1)$ and $\tilde p_\xi(0) = 0$. By integrating both sides of the differential equation, we get that the time T_ξ taken by p_ξ to reach the final state p_T reads

$$T_\xi = \int_0^{p_\xi(\xi)} \frac{d\nu}{f(\nu) + Mg(\nu)} + \int_{p_\xi(\xi)}^{p_T} \frac{d\nu}{f(\nu)}.$$

Note that in case $p_\xi(\xi) \leqslant \theta$ the second integral does not converge unless $\xi = 1$, in which case it vanishes. This expression gives a lower bound on C depending on p_T. If $p_T \leqslant \theta$ we must have $\xi = 1$, concluding that $p_\xi(\xi) = p_T$ and $C \geqslant MT^{p_T} = C^{p_T}$, with C^{p_T} as defined in (18). Instead if $p_T > \theta$, then either $\xi = 1$ implying $C \geqslant C^{p_T}$ or $p_\xi(\xi) > \theta$ and therefore $C > M \int_0^\theta \frac{d\nu}{f(\nu)+Mg(\nu)} = C^\theta$. Since in this case $C^{p_T} > C^\theta$, the least restrictive condition is $C > C^\theta$. We conclude that under the hypothesis of this proposition $T_\xi < \infty$ and $\mathcal{D}^{p_T}$ is non-empty.

Let us consider a minimizing sequence $(T_n, \tilde{u}_n)_{n\in\mathbb{N}} \in (\mathcal{D}^{p_T})^{\mathbb{N}}$ and let $\tilde{p}_n$ be the solution of $\tilde{p}' = T\left(f(\tilde{p}) + \tilde{u}_n g(\tilde{p}_n)\right)$ in $(0,1)$ and $\tilde{p}_n(0) = 0$. By minimality, one has $\lim\limits_{n\to\infty} \tilde{J}_\alpha(T_n, \tilde{u}_n) < \infty$, i.e.

$$\lim_{n\to\infty} (1-\alpha)\, T_n \int_0^1 j_1(\tilde{u}_n(s))ds + \alpha j_2(T_n, p_T) < \infty.$$

Each term of the sum being bounded from below by 0, it follows that both of them are also bounded above. Since $\alpha > 0$ and $\lim_{n\to\infty} j_2(T_n, p_T) = +\infty$, it follows that $(T_n)_{n\in\mathbb{N}}$ is bounded, and therefore, up to a subsequence, $T_n \to \tilde{T} < \infty$ as $n \to +\infty$. By mimicking the arguments used for problem $(\mathcal{P}^{2,\alpha}_{T,C,M})$, one shows that, up to a subsequence, $(\tilde{u}_n)_{n\in\mathbb{N}}$ converges to $\tilde{u}^* \in \mathcal{U}_{\tilde{T},C,M}$ weakly-star in $L^\infty(0,1;[0,M])$. Moreover,$(\tilde{p}_n)_{n\in\mathbb{N}}$ converges to $\tilde{p}^*$ in $\mathcal{C}^0([0,\tilde{T}])$, where $\tilde{p}^*$ solves the equation

$$(\tilde{p}^*)' = \tilde{T}(f(\tilde{p}^*) + \tilde{u}^* g(\tilde{p}^*)) \quad \text{in } (0,\tilde{T})$$

and $\tilde{p}^*(0) = 0$. As a consequence, $(\tilde{J}_\alpha(T_n, \tilde{u}_n))_{n\in\mathbb{N}}$ converges to $\tilde{J}_\alpha(\tilde{T}, \tilde{u}^*)$, which concludes the proof. □

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